Fundamentals of Applied Electromagnetics 6e

by

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Exercise Solutions

Chapters

Chapter 1 Exercise Solutions

Exercise 1.1 Consider the red wave shown in Fig. E1.1. What is the wave's (a) amplitude, (b) wavelength, and (c) frequency, given that its phase velocity is 6 m/s?

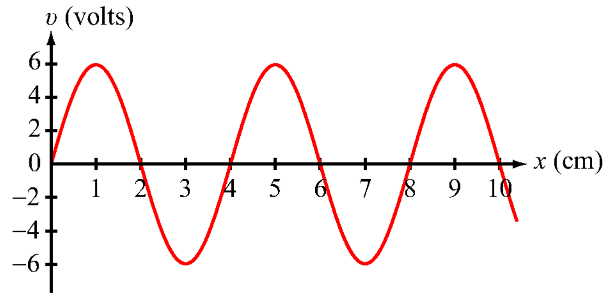


Figure E1.1

Solution:

(a) $A = 6$ V.

(b) $\lambda = 4$ cm.

(c) $f = \dfrac{u_p}{\lambda} = \dfrac{6}{4 \times 10^{-2}} = 150$ Hz.

Exercise 1.2 The wave shown in red in Fig. E1.2 is given by $\upsilon = 5\cos 2\pi t/8$. Of the following four equations:

(1) $\upsilon = 5\cos(2\pi t/8 - \pi/4)$,

(2) $\upsilon = 5\cos(2\pi t/8 + \pi/4)$,

(3) $\upsilon = -5\cos(2\pi t/8 - \pi/4)$,

(4) $\upsilon = 5\sin 2\pi t/8$,

(a) which equation applies to the green wave? (b) which equation applies to the blue wave?

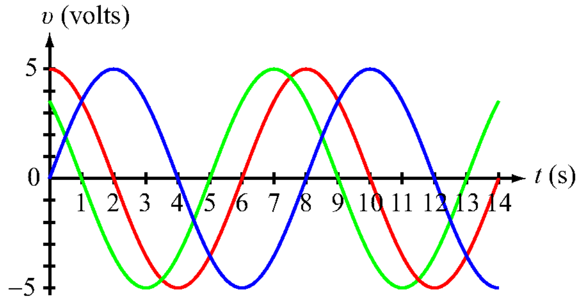


Figure E1.2

Solution:

(a) The green wave has an amplitude of 5 V and a period $T = 8$ s. Its peak occurs earlier than that of the red wave; hence, its constant phase angle is positive relative to that of the red wave. A full cycle of 8 s corresponds to 2π in phase. The green wave crosses the time axis 1 s sooner than the red wave. Hence, its phase angle is

$$\phi_0 = \frac{1}{8} \times 2\pi = \frac{\pi}{4} .$$

Consequently,

$$\begin{aligned} \upsilon &= 5\cos(2\pi t/T + \phi_0) \\ &= 5\cos(2\pi t/7 + \pi/4), \end{aligned}$$

which is given by #2.

(b) The blue wave's period $T = 8$ s. Its phase angle is delayed relative to the red wave by 2 s. Hence, the phase angle is negative and given by

$$\phi_0 = -\frac{2}{8} \times 2\pi = -\frac{\pi}{2} ,$$

and

$$\begin{aligned} \upsilon &= 5\cos\left(\frac{2\pi t}{8} - \frac{\pi}{2}\right) \\ &= 5\sin 2\pi t/8, \end{aligned}$$

which is given by #4.

Exercise 1.3 The electric field of a traveling electromagnetic wave is given by

$$E(z,t) = 10\cos(\pi \times 10^7 t + \pi z/15 + \pi/6) \quad \text{(V/m)}.$$

Determine (a) the direction of wave propagation, (b) the wave frequency f, (c) its wavelength λ, and (d) its phase velocity u_p.

Solution:

(a) $-z$-direction because the signs of the coefficients of t and z are both positive.

(b) From the given expression,

$$\omega = \pi \times 10^7 \quad \text{(rad/s)}.$$

Hence,

$$f = \frac{\omega}{2\pi} = \frac{\pi \times 10^7}{2\pi} = 5 \times 10^6 \text{ Hz} = 5 \text{ MHz}.$$

(c) From the given expression,

$$\frac{2\pi}{\lambda} = \frac{\pi}{15}.$$

Hence $\lambda = 30$ m.

(d) $u_p = f\lambda = 5 \times 10^6 \times 30 = 1.5 \times 10^8$ m/s.

Exercise 1.4 Consider the red wave shown in Fig. E1.4. What is the wave's (a) amplitude (at $x = 0$), (b) wavelength, and (c) attenuation constant?

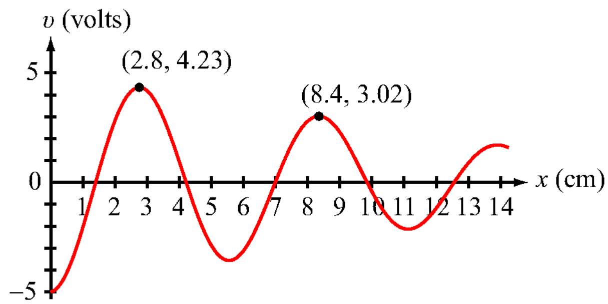


Figure E1.4

Solution: The wave shown in the figure exhibits a sinusoidal variation in x and its amplitude decreases as a function of x. Hence, it can be described by the general expression

$$\upsilon = Ae^{-\alpha x}\cos\left(\frac{2\pi x}{\lambda} + \phi_0\right).$$

From the given coordinates of the first two peaks, we deduce that

$$\lambda = 8.4 - 2.8 = 5.6 \text{ cm}.$$

At $x = 0$, $\upsilon = -5$ V and it occurs exactly $\lambda/2$ before the first peak. Hence, the wave amplitude is 5 V, and from

$$-5 = 5\cos(0 + \phi_0),$$

it follows that

$$\phi_0 = \pi.$$

Consequently,

$$\upsilon = 5e^{-\alpha x}\cos\left(\frac{2\pi x}{5.6} + \pi\right).$$

In view of the relation $\cos x = -\cos(x \pm \pi)$, υ can be expressed as

$$\upsilon = -5e^{-\alpha x}\cos\frac{2\pi x}{5.6} \quad \text{(V)}.$$

We can describe the amplitude as 5 V for a wave with a constant phase angle of π, or as -5 V with a phase angle of zero.

At $x = 2.8$ cm,

$$\upsilon(x = 2.8) = 4.23 = -5e^{-2.8\alpha}\cos\left(\frac{2\pi \times 2.8}{5.6}\right)$$
$$= 5e^{-2.8\alpha}.$$

Hence,

$$e^{-2.8\alpha} = \frac{4.23}{5} \,,$$

and

$$\alpha = -\frac{1}{2.8} \ln\left(\frac{4.23}{5}\right) = 0.06 \text{ Np/cm.}$$

Exercise 1.5 The red wave shown in Fig. E1.5 is given by $\upsilon = 5\cos 4\pi x$ (V). What expression is applicable to (a) the blue wave and (b) the green wave?

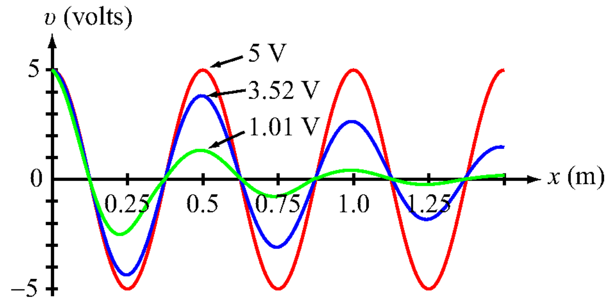


Figure E1.5

Solution: At $x = 0$, all three waves start at their peak value of 5 V. Also, $\lambda = 0.5$ m for all three waves. Hence, they share the general form

$$\upsilon = Ae^{-\alpha x}\cos\frac{2\pi x}{\lambda}$$
$$= 5e^{-\alpha x}\cos 4\pi x \quad \text{(V)}.$$

For the red wave, $\alpha = 0$.
For the blue wave,

$$3.52 = 5e^{-0.5\alpha} \quad \Rightarrow \quad \alpha = 0.7 \text{ Np/m}.$$

For the green wave,

$$1.01 = 5e^{-0.5\alpha} \quad \Rightarrow \quad \alpha = 3.2 \text{ Np/m}.$$

Exercise 1.6 An electromagnetic wave is propagating in the z-direction in a lossy medium with attenuation constant $\alpha = 0.5$ Np/m. If the wave's electric-field amplitude is 100 V/m at $z = 0$, how far can the wave travel before its amplitude will have been reduced to (a) 10 V/m, (b) 1 V/m, (c) 1 μV/m?

Solution:

(a)

$$\begin{aligned}
100e^{-\alpha z} &= 10 \\
100e^{-0.5z} &= 10 \\
e^{-0.5z} &= 0.1 \\
-0.5z &= \ln 0.1 = -2.3 \\
z &= 4.6 \text{ m.}
\end{aligned}$$

(b)

$$\begin{aligned}
100e^{-0.5z} &= 1 \\
z &= \frac{\ln 0.01}{-0.5} = 9.2 \text{ m.}
\end{aligned}$$

(c)

$$\begin{aligned}
100e^{-0.5z} &= 10^{-6} \\
z &= \frac{\ln 10^{-8}}{-0.5} = 37 \text{ m.}
\end{aligned}$$

Exercise 1.7 Express the following complex functions in polar form:

$$z_1 = (4 - j3)^2,$$
$$z_2 = (4 - j3)^{1/2}.$$

Solution:

$$\begin{aligned} z_1 &= (4 - j3)^2 \\ &= \left[(4^2 + 3^2)^{1/2} \angle{-\tan^{-1} 3/4}\right]^2 \\ &= [5\angle{-36.87^\circ}]^2 = 25\angle{-73.7^\circ}. \end{aligned}$$

$$\begin{aligned} z_2 &= (4 - j3)^{1/2} \\ &= \left[(4^2 + 3^2)^{1/2} \angle{-j\tan^{-1} 3/4}\right]^{1/2} \\ &= [5\angle{-36.87^\circ}]^{1/2} = \pm\sqrt{5}\angle{-18.4^\circ}. \end{aligned}$$

Exercise 1.8 Show that $\sqrt{2j} = \pm(1+j)$.

Solution:

$$e^{j\pi/2} = 0 + j\sin(\pi/2) = j$$

$$\begin{aligned}
\sqrt{2j} = [2e^{j\pi/2}]^{1/2} &= \pm\sqrt{2}\, e^{j\pi/4} \\
&= \pm\sqrt{2}(\cos\pi/4 + j\sin\pi/4) \\
&= \pm\sqrt{2}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) \\
&= \pm(1+j).
\end{aligned}$$

Exercise 1.9 A series RL circuit is connected to a voltage source given by $v_s(t) = 150\cos\omega t$ (V). Find (a) the phasor current $\tilde{I}$ and (b) the instantaneous current $i(t)$ for $R = 400\ \Omega$, $L = 3$ mH, and $\omega = 10^5$ rad/s.

Solution:

(a) From Example 1–4,

$$\begin{aligned}
\tilde{I} &= \frac{\widetilde{V}_s}{R + j\omega L} \\
&= \frac{150}{400 + j10^5 \times 3 \times 10^{-3}} \\
&= \frac{150}{400 + j300} = 0.3\angle{-36.9^\circ} \quad \text{(A)}.
\end{aligned}$$

(b)

$$\begin{aligned}
i(t) &= \mathfrak{Re}[\widetilde{I}e^{j\omega t}] \\
&= \mathfrak{Re}[0.3e^{-j36.9^\circ}e^{j10^5 t}] \\
&= 0.3\cos(10^5 t - 36.9^\circ) \quad \text{(A)}.
\end{aligned}$$

Exercise 1.10 A phasor voltage is given by $\widetilde{V} = j5$ V. Find $v(t)$.

Solution:

$$\begin{aligned}
\widetilde{V} &= j5 = 5e^{j\pi/2} \\
v(t) &= \mathfrak{Re}[\widetilde{V} e^{j\omega t}] \\
&= \mathfrak{Re}[5e^{j\pi/2} e^{j\omega t}] \\
&= 5\cos\left(\omega t + \frac{\pi}{2}\right) = -5\sin\omega \quad \text{(V)}.
\end{aligned}$$

Chapter 2 Exercise Solutions

Exercise 2.1 Use Table 2-1 to compute the line parameters of a two-wire air line whose wires are separated by a distance of 2 cm, and each is 1 mm in radius. The wires may be treated as perfect conductors with $\sigma_c = \infty$.

Solution: Two-wire air line: Because medium between wires is air, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ and $\sigma = 0$.

$$d = 2 \text{ cm}, \qquad a = 1 \text{ mm}, \qquad \sigma_c = \infty$$

$$R_s = \left[\frac{\pi f \mu_c}{\sigma_c}\right]^{1/2} = 0$$

$$R' = 0$$

$$\begin{aligned} L' &= \frac{\mu_0}{\pi} \ln\left[\left(\frac{d}{2a}\right) + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right] \\ &= \left(\frac{4\pi \times 10^{-7}}{\pi}\right) \ln\left[\left(\frac{20}{2}\right) + \sqrt{\left(\frac{20}{2}\right)^2 - 1}\right] \\ &= 4 \times 10^{-7} \ln[10 + \sqrt{99}] = 1.2 \quad (\mu\text{H/m}). \end{aligned}$$

$$G' = 0 \qquad \text{because } \sigma = 0$$

$$C' = \frac{\pi \varepsilon_0}{\ln\left[\left(\frac{d}{2a}\right) + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right]} = \frac{\pi \times 8.85 \times 10^{-12}}{\ln[10 + \sqrt{99}]} = 9.29 \quad (\text{pF/m}).$$

Exercise 2.2 Calculate the transmission line parameters at 1 MHz for a rigid coaxial air line with an inner conductor diameter of 0.6 cm and an outer conductor diameter of 1.2 cm. The conductors are made of copper [see Appendix B for μ_c and σ_c of copper].

Solution: Coaxial air line: Because medium between wires is air, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ and $\sigma = 0$.

$$a = 0.3 \text{ cm}, \qquad b = 0.6 \text{ cm}, \qquad \mu_c = \mu_0, \qquad \sigma_c = 5.8 \times 10^7 \text{ S/m}$$

$$\begin{aligned} R_s &= \sqrt{\pi f \mu_c / \sigma_c} \\ &= [\pi \times 10^6 \times 4\pi \times 10^{-7} / (5.8 \times 10^7)]^{1/2} = 2.6 \times 10^{-4} \ \Omega. \end{aligned}$$

$$R' = \frac{R_s}{2\pi} \left(\frac{1}{a} + \frac{1}{b} \right) = \frac{2.6 \times 10^{-4}}{2\pi} \left(\frac{1}{3 \times 10^{-3}} + \frac{1}{6 \times 10^{-3}} \right) = 2.08 \times 10^{-2} \quad (\Omega/\text{m})$$

$$L' = \frac{\mu_0}{2\pi} \ln \left(\frac{b}{a} \right) = \frac{4\pi \times 10^{-7}}{2\pi} \ln 2 = 0.14 \quad (\mu\text{H/m})$$

$$G' = 0 \qquad \text{because } \sigma = 0$$

$$C' = \frac{2\pi\varepsilon}{\ln(b/a)} = \frac{2\pi \times 8.85 \times 10^{-12}}{\ln 2} = 80.3 \quad (\text{pF/m}).$$

Exercise 2.3 Verify that Eq. (2.26a) is indeed a solution of the wave equation given by Eq. (2.21).

Solution:

$$\widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\frac{d^2\,\widetilde{V}(z)}{dz^2} - \gamma^2\,\widetilde{V}(z) \stackrel{?}{=} 0$$

$$\frac{d^2}{dz^2}(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}) - \gamma^2 (V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}) \stackrel{?}{=} 0$$

$$\gamma^2 V_0^+ e^{-\gamma z} + \gamma^2 V_0^- e^{\gamma z} - \gamma^2 V_0^+ e^{-\gamma z} - \gamma^2 V_0^- e^{\gamma z} = 0.$$

Exercise 2.4 A two-wire air line has the following line parameters: $R' = 0.404$ (mΩ/m), $L' = 2.0$ (μH/m), $G' = 0$, and $C' = 5.56$ (pF/m). For operation at 5 kHz, determine (a) the attenuation constant α, (b) the phase constant β, (c) the phase velocity u_p, and (d) the characteristic impedance Z_0.

Solution: Given:

$$
\begin{aligned}
R' &= 0.404 \text{ (m}\Omega\text{/m)}, & G' &= 0,\\
L' &= 2.0 \text{ (}\mu\text{H/m)}, & C' &= 5.56 \text{ (pF/m)}.
\end{aligned}
$$

(a)

$$
\begin{aligned}
\alpha &= \mathfrak{Re}\left\{[(R' + j\omega L')(G' + j\omega C')]^{1/2}\right\}\\
&= \mathfrak{Re}\left\{[(0.404 \times 10^{-3} + j2\pi \times 5 \times 10^3 \times 2 \times 10^{-6})(0 + j2\pi \times 5 \times 10^3 \times 5.56 \times 10^{-12})]^{1/2}\right\}\\
&= \mathfrak{Re}[3.37 \times 10^{-7} + j1.05 \times 10^{-4}]\\
\alpha &= 3.37 \times 10^{-7} \quad \text{(Np/m)}.
\end{aligned}
$$

(b) From part (a),

$$
\begin{aligned}
\beta &= \mathfrak{Im}\left\{[(R' + j\omega L')(G' + j\omega C')]^{1/2}\right\}\\
&= 1.05 \times 10^{-4} \quad \text{(rad/m)}.
\end{aligned}
$$

(c)

$$
u_p = \frac{\omega}{\beta} = \frac{2\pi \times 5 \times 10^3}{1.05 \times 10^{-4}} = 3 \times 10^8 \quad \text{(m/s)}.
$$

(d)

$$
\begin{aligned}
Z_0 &= \frac{R' + j\omega L'}{\alpha + j\beta}\\
&= \frac{0.404 \times 10^{-3} + j5 \times 10^3 \times 2 \times 10^{-6}}{3.37 \times 10^{-7} + j1.05 \times 10^{-4}}\\
&= (600 - j2)\ \Omega.
\end{aligned}
$$

Exercise 2.5 For a lossless transmission line, $\lambda = 20.7$ cm at 1 GHz. Find ε_r of the insulating material.

Solution:

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

$$\varepsilon_r = \left(\frac{\lambda_0}{\lambda}\right)^2 = \left(\frac{c}{f\lambda}\right)^2 = \left(\frac{3\times10^8}{1\times10^8\times20.7\times10^{-2}}\right)^2 = 2.1.$$

Exercise 2.6 A lossless transmission line uses a dielectric insulating material with $\varepsilon_r = 4$. If its line capacitance is $C' = 10$ (pF/m), find (a) the phase velocity u_p, (b) the line inductance L', and (c) the characteristic impedance Z_0.

Solution:

(a)

$$u_p = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times10^8}{\sqrt{4}} = 1.5\times10^8 \text{ m/s}.$$

(b)

$$u_p = \frac{1}{\sqrt{L'C'}}, \qquad u_p^2 = \frac{1}{L'C'}.$$

$$L' = \frac{1}{u_p^2 C'} = \frac{1}{(1.5\times10^8)^2\times10\times10^{-12}} = 4.45 \quad (\mu\text{H/m}).$$

(c)

$$Z_0 = \sqrt{\frac{L'}{C'}} = \left(\frac{4.45\times10^{-6}}{10\times10^{-12}}\right)^{1/2} = 667.1\ \Omega.$$

Exercise 2.7 A 50-Ω lossless transmission line is terminated in a load impedance $Z_L = (30 - j200)$ Ω. Calculate the voltage reflection coefficient at the load.

Solution:

$$\begin{aligned}\Gamma &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= \frac{30 - j200 - 50}{(30 - j200) + 50} = \frac{-20 - j200}{80 - j200} = 0.93\angle{-27.5^\circ}.\end{aligned}$$

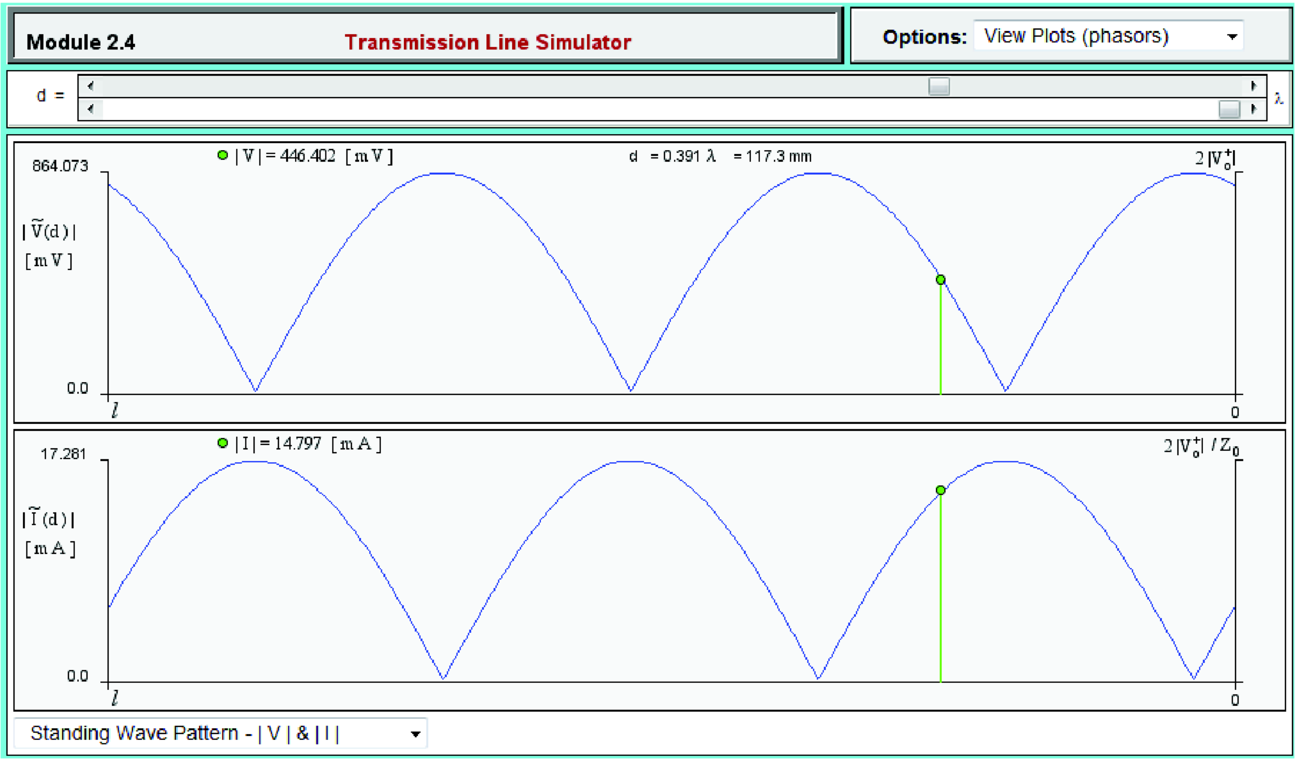
Exercise 2.8 A 150-Ω lossless line is terminated in a capacitor whose impedance is $Z_L = -j30\ \Omega$. Calculate Γ.

Solution:

$$\begin{aligned}\Gamma &= \frac{Z_L - Z_0}{Z_L + Z_0}\\ &= \frac{-j30 - 150}{-j30 + 150} = 1\angle{-157.4^\circ}.\end{aligned}$$

Exercise 2.9 Use CD Module 2.4 to generate the voltage and current standing-wave patterns for a 50-Ω line of length 1.5λ, terminated in an inductance with $Z_L = j140\ \Omega$.

Solution: Standing-wave patterns generated with the help of DVD Module 2.4 are shown.



Exercise 2.10 If $\Gamma = 0.5\angle{-60^\circ}$ and $\lambda = 24$ cm, find the locations of the voltage maximum and minimum nearest to the load.

Solution:

$$\Gamma = 0.5\angle{-60^\circ}\,, \qquad \lambda = 24 \text{ cm}$$

$$\begin{aligned} l_{\text{max}} &= \frac{\theta_{\text{r}}\lambda}{4\pi} + \frac{\lambda}{2} \qquad \text{(because } \theta_{\text{r}} \text{ is negative)} \\ &= \left[\frac{(-\pi/3)\times 24}{4\pi} + \frac{24}{2}\right] \text{ cm} = [-2+12] \text{ cm} = 10 \text{ cm.} \end{aligned}$$

$$\begin{aligned} l_{\text{min}} &= l_{\text{max}} - \frac{\lambda}{4} \qquad \text{(because } l_{\text{max}} > \lambda/4\text{)} \\ &= \left(10 - \frac{24}{4}\right) \text{ cm} = 4 \text{ cm.} \end{aligned}$$

Exercise 2.11 A 140-Ω lossless line is terminated in a load impedance $Z_L = (280 + j182)$ Ω. If $\lambda = 72$ cm, find (a) the reflection coefficient Γ, (b) the voltage standing-wave ratio S, (c) the locations of voltage maxima, and (d) the locations of voltage minima.

Solution:

$$Z_0 = 140\ \Omega, \qquad Z_L = (280 + j182)\ \Omega$$

(a)

$$\begin{aligned}\Gamma &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= \frac{280 + j182 - 140}{280 + j182 + 140} = \frac{140 + j182}{420 + j182} = 0.5\angle 29^\circ .\end{aligned}$$

(b)

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.5}{1 - 0.5} = \frac{1.5}{0.5} = 3.$$

(c)

$$\begin{aligned}l_{\max} &= \frac{\theta_r \lambda}{4\pi} + \frac{n\lambda}{2}, \qquad n = 0, 1, 2, \ldots \\ &= \frac{(29\pi/180) \times 0.72}{4\pi} + \frac{n \times 0.72}{2} \\ &= (2.9 + 36n) \quad \text{(cm)}, \qquad n = 0, 1, 2, \ldots\end{aligned}$$

(d)

$$\begin{aligned}l_{\min} &= l_{\max} + \frac{\lambda}{4} \\ &= \left[(2.9 + 36n) + \frac{72}{4}\right] \text{ cm} \\ &= (20.9 + 36n) \text{ cm}, \qquad n = 0, 1, 2, \ldots\end{aligned}$$

Exercise 2.12 A 50-Ω lossless transmission line uses an insulating material with $\varepsilon_r = 2.25$. When terminated in an open circuit, how long should the line be for its input impedance to be equivalent to a 10-pF capacitor at 50 MHz?

Solution: For a 10-pF capacitor at 50 MHz,

$$Z_c = \frac{1}{j\omega C} = \frac{-j}{2\pi \times 50 \times 10^6 \times 10 \times 10^{-12}} = -j\frac{1000}{\pi}\ \Omega$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi\sqrt{\varepsilon_r}}{\lambda_0} = \frac{2\pi f\sqrt{\varepsilon_r}}{c}$$

$$= \frac{2\pi \times 5 \times 10^7\sqrt{2.25}}{3 \times 10^8} = 1.57 \quad \text{(rad/m)}.$$

For lossless lines with open-circuit termination,

$$Z_{in} = -jZ_0 \cot\beta l = -j50\cot 1.57l$$

Hence,

$$-j\frac{1000}{\pi} = -j50\cot 1.57l$$

or

$$l = 9.92 \quad \text{(cm)}.$$

Exercise 2.13 A 300-Ω feedline is to be connected to a 3-m long, 150-Ω line terminated in a 150-Ω resistor. Both lines are lossless and use air as the insulating material, and the operating frequency is 50 MHz. Determine (a) the input impedance of the 3-m long line, (b) the voltage standing-wave ratio on the feedline, and (c) the characteristic impedance of a quarter-wave transformer were it to be used between the two lines in order to achieve $S = 1$ on the feedline.

Solution: At 50 MHz,

$$\lambda = \lambda_0 = \frac{c}{f} = \frac{3\times 10^8}{5\times 10^7} = 6 \text{ m}.$$

(a)

$$\frac{l}{\lambda} = \frac{3}{6} = 0.5.$$

Hence, $Z_{\text{in}} = Z_{\text{L}} = 150\ \Omega$. ($Z_{\text{in}} = Z_{\text{L}}$ if $Z = n\lambda/2$.)

(b)

$$\Gamma = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \frac{150-300}{150+300} = \frac{-150}{450} = -\frac{1}{3}.$$

$$S = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = \frac{4/3}{2/3} = 2.$$

(c)

$$Z_{02}^2 = Z_1 Z_3 = 300\times 150 = 45,000$$
$$Z_{02} = 212.1\ \Omega.$$

where Z_1 is the feedline and Z_3 is Z_{in} of part (a).

Exercise 2.14 For a 50-Ω lossless transmission line terminated in a load impedance $Z_L = (100 + j50)\ \Omega$, determine the fraction of the average incident power reflected by the load.

Solution:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$
$$= \frac{100 + j50 - 50}{100 + j50 + 50} = \frac{50 + j50}{150 + j50} = 0.45\angle 26.6^\circ .$$

Fraction of reflected power $= |\Gamma|^2 = (0.45)^2 = 20\%$.

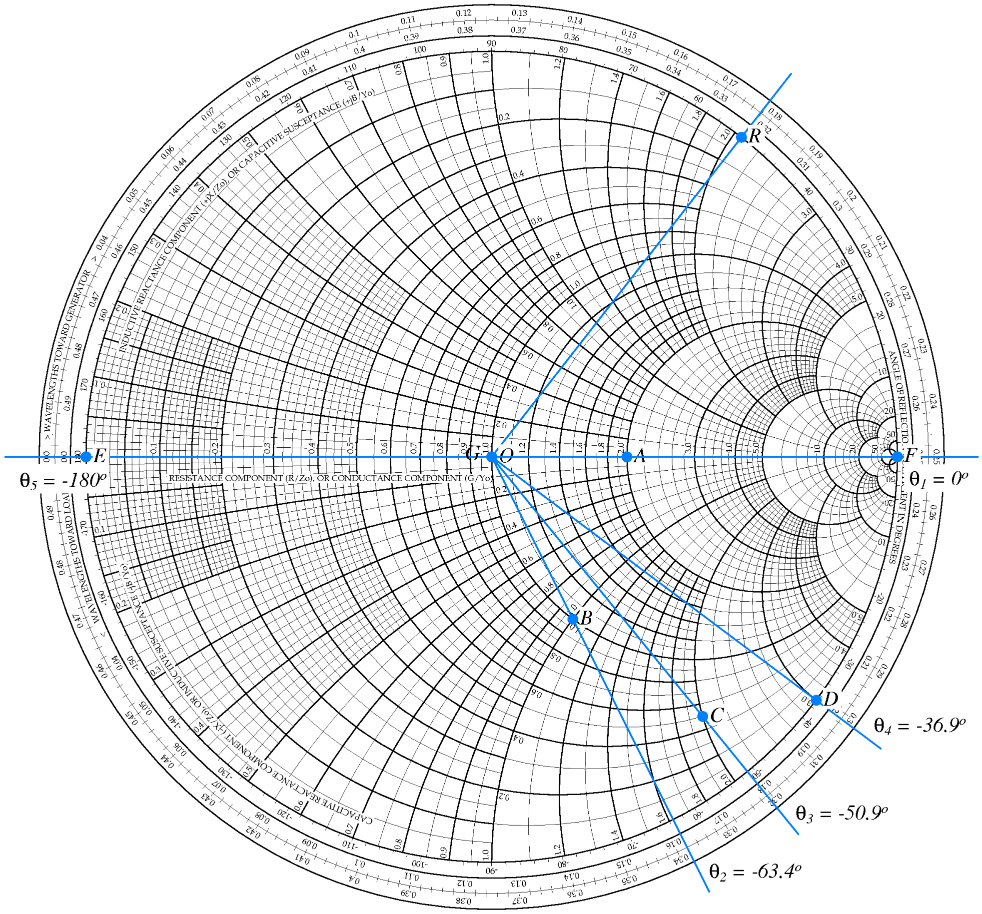
Exercise 2.15 For the line of Exercise 2.14, what is the magnitude of the average reflected power if $|V_0^+| = 1$ V?

Solution:

$$P_{\text{av}}^{\text{r}} = |\Gamma|^2 \frac{|V_0^+|^2}{2Z_0} = \frac{0.2 \times 1}{2 \times 50} = 2 \quad \text{(mW)}.$$

Exercise 2.16 Use the Smith chart to find the values of Γ corresponding to the following normalized load impedances: (a) $z_L = 2 + j0$, (b) $z_L = 1 - j1$, (c) $z_L = 0.5 - j2$, (d) $z_L = -j3$, (e) $z_L = 0$ (short circuit), (f) $z_L = \infty$ (open circuit), (g) $z_L = 1$ (matched load).

Solution:



(a)

$$\Gamma = \frac{OA}{OR}\angle\theta_1 = 0.33$$

(b)

$$\Gamma = \frac{OB}{OR}\angle\theta_2 = 0.45\angle -63.4^\circ$$

(c)

$$\Gamma = \frac{OC}{OR}\angle\theta_3 = 0.83\angle -50.9^\circ$$

(d)

$$\Gamma = \frac{OD}{OR}\angle\theta_4 = 1\angle -36.9^\circ$$

(e)

$$\Gamma = \frac{OE}{OR} \angle \theta_5 = 1 \angle 180^\circ = -1$$

(f)

$$\Gamma = \frac{OF}{OR} \angle \theta_1 = 1$$

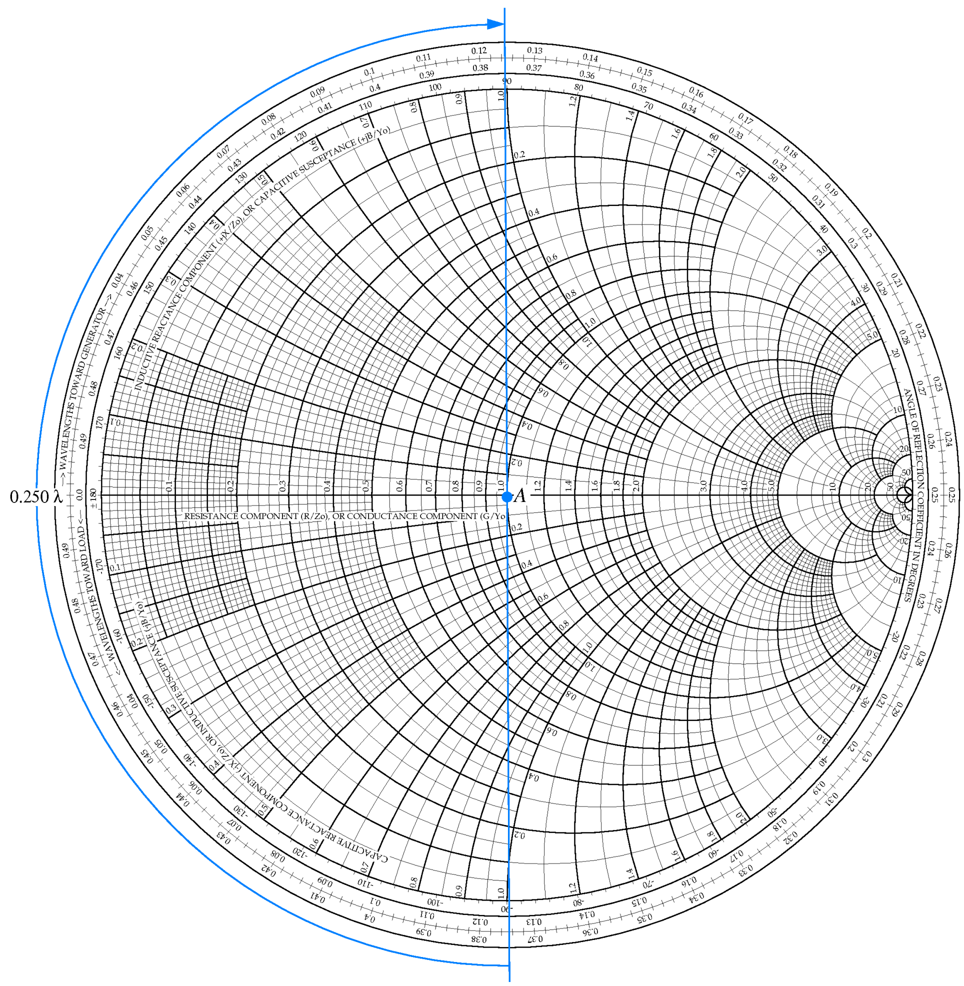
(g)

$$\Gamma = \frac{OG}{OR} = 0$$

Exercise 2.17 Use the Smith chart to find the normalized input impedance of a lossless line of length l terminated in a normalized load impedance z_L for each of the following combinations: (a) $l = 0.25\lambda$, $z_L = 1 + j0$, (b) $l = 0.5\lambda$, $z_L = 1 + j1$, (c) $l = 0.3\lambda$, $z_L = 1 - j1$, (d) $l = 1.2\lambda$, $z_L = 0.5 - j0.5$, (e) $l = 0.1\lambda$, $z_L = 0$ (short circuit), (f) $l = 0.4\lambda$, $z_L = j3$, (g) $l = 0.2\lambda$, $z_L = \infty$ (open circuit).

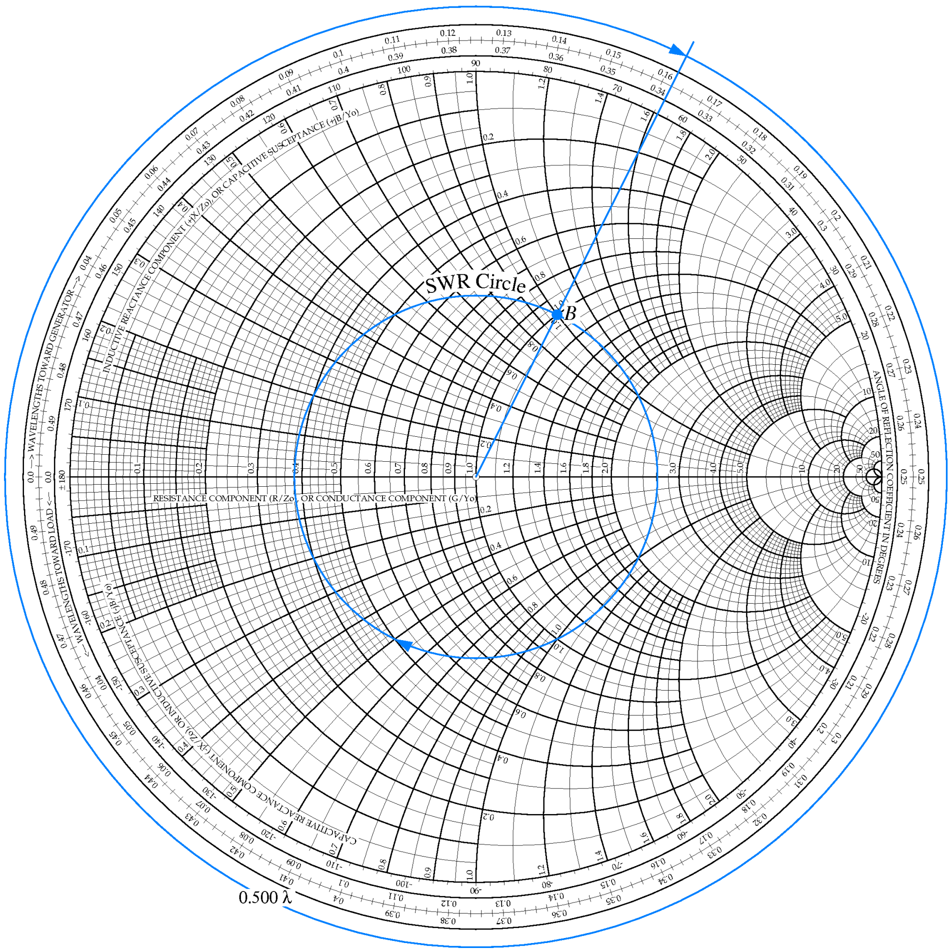
Solution:

(a)



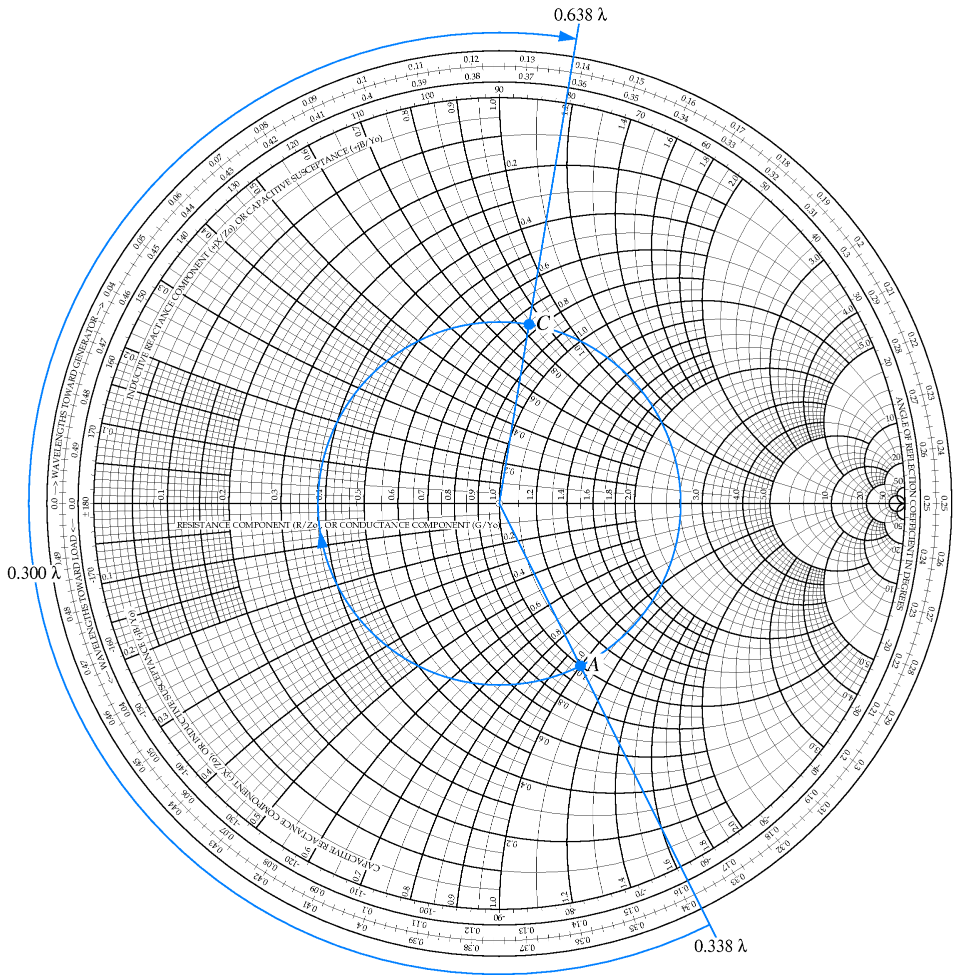
$z_{in} = 1 + j0$

(b)



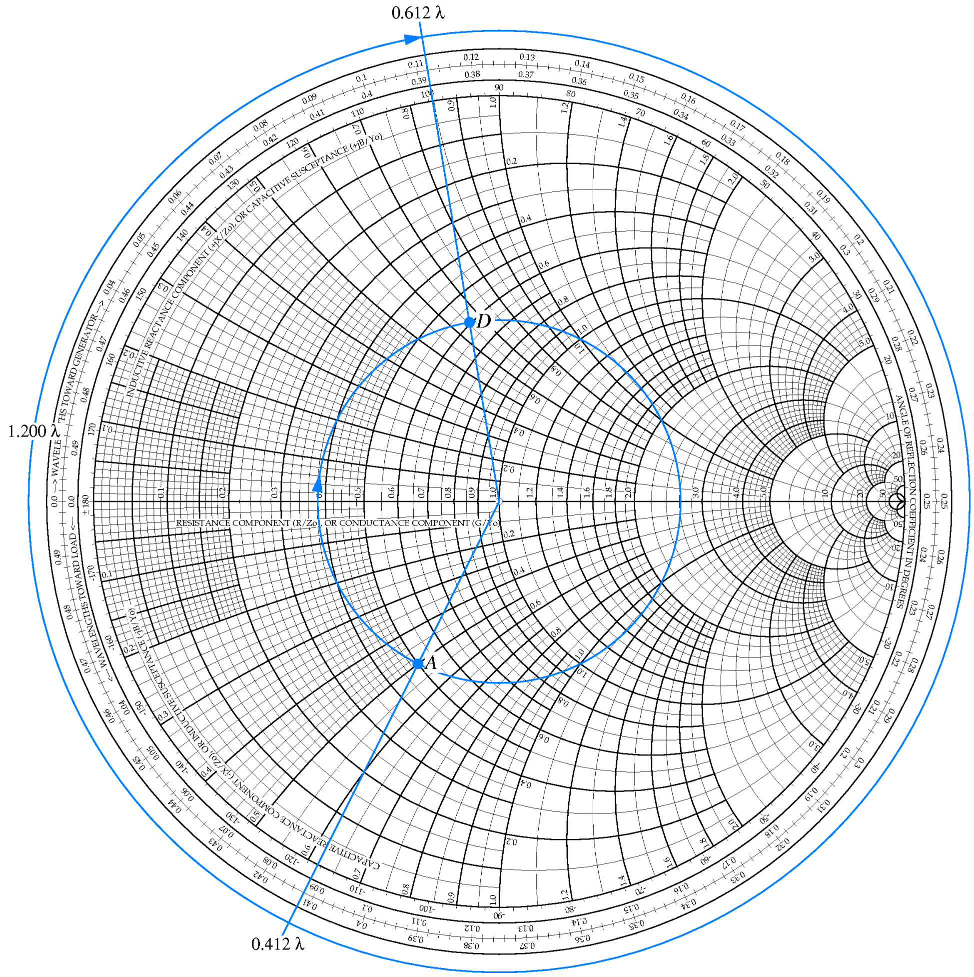
$z_{in} = 1 + j1$

(c)



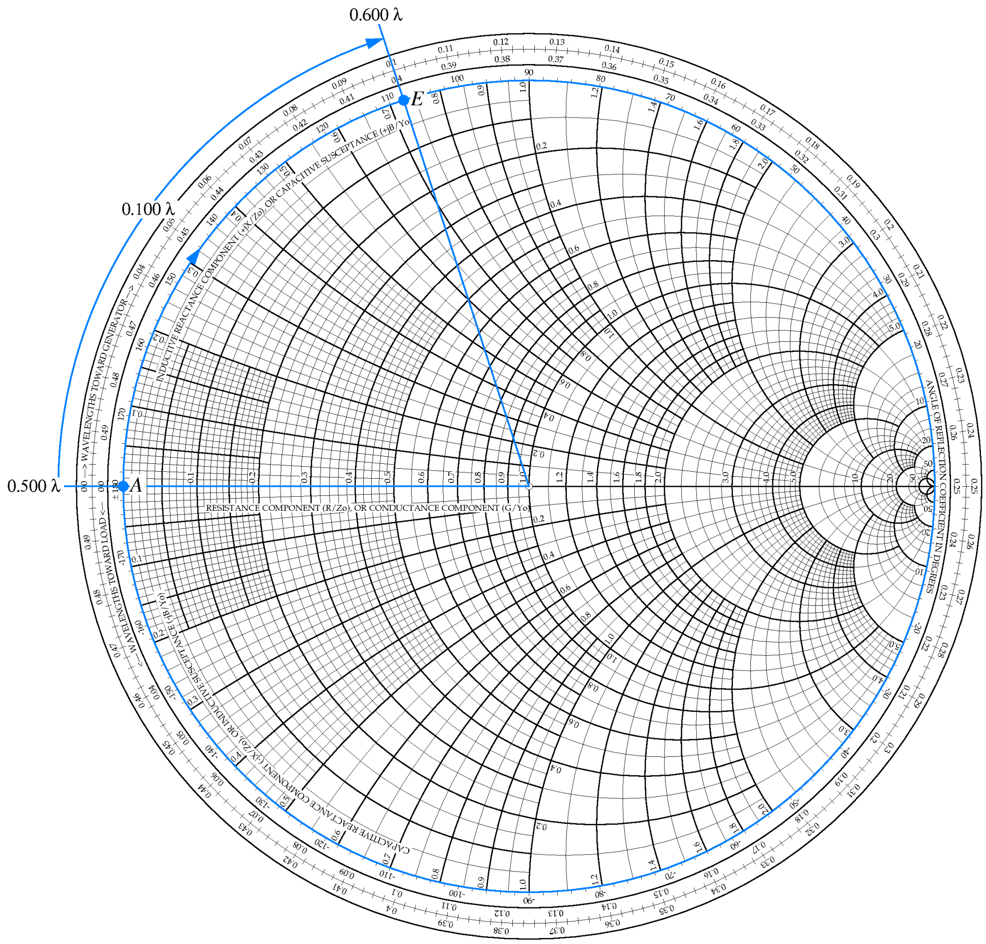
$z_{\text{in}} = 0.76 + j0.84$

(d)



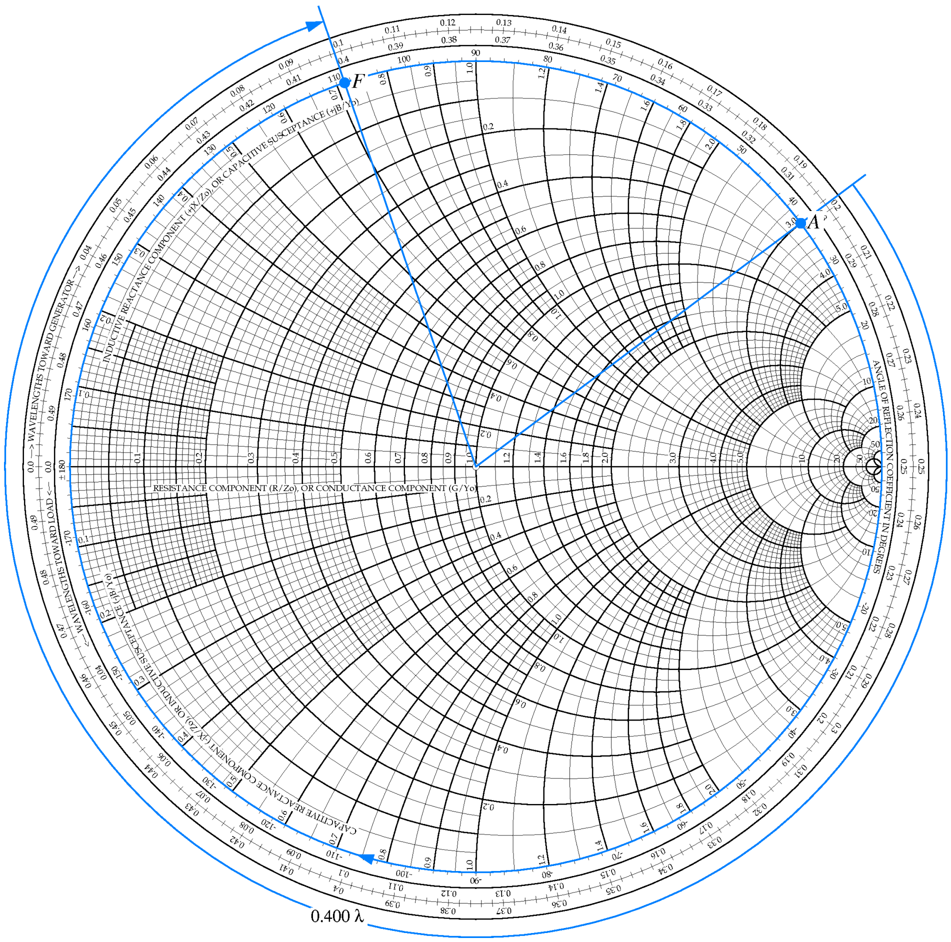
$z_{\text{in}} = 0.59 + j0.66$

(e)



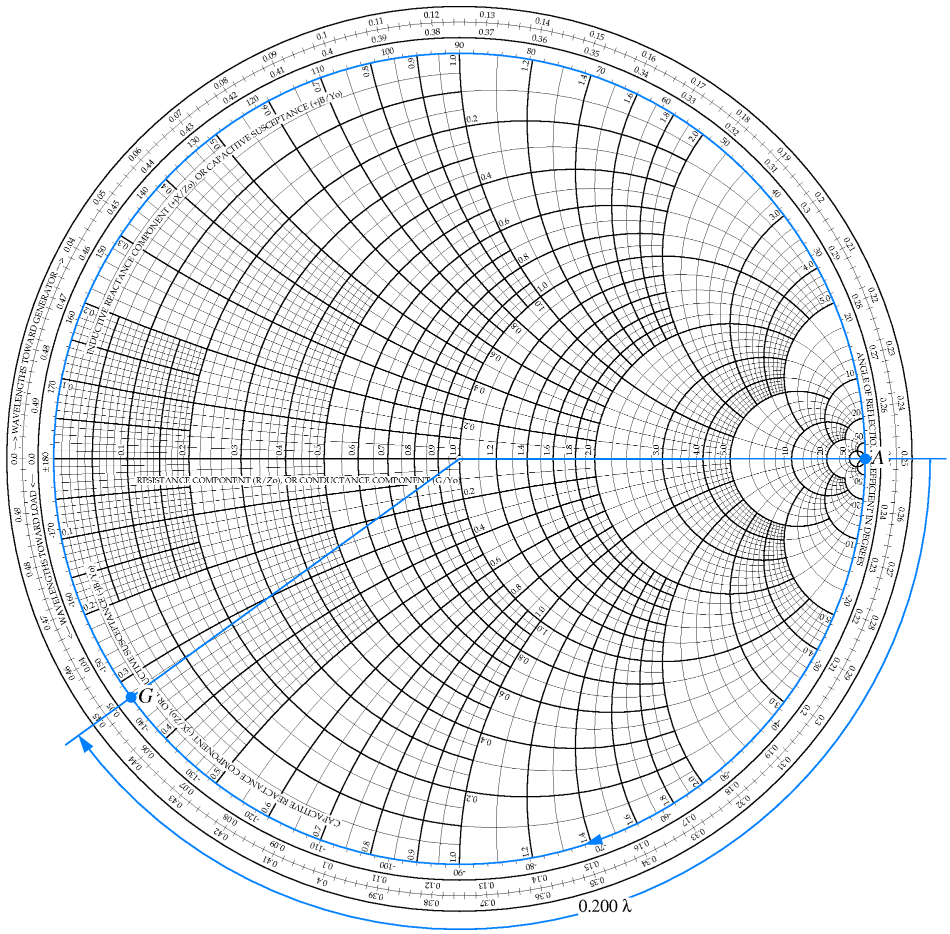
$z_{\text{in}} = 0 + j0.73$

(f)



$z_{\text{in}} = 0 + j0.72$

(g)



$z_{in} = 0 - j0.32$

Chapter 3 Exercise Solutions

Exercise 3.1 Find the distance vector between $P_1(1,2,3)$ and $P_2(-1,-2,3)$ in Cartesian coordinates.

Solution:

$$\begin{aligned}\overrightarrow{P_1P_2} &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1) \\ &= \hat{\mathbf{x}}(-1-1) + \hat{\mathbf{y}}(-2-2) + \hat{\mathbf{z}}(3-3) \\ &= -\hat{\mathbf{x}}2 - \hat{\mathbf{y}}4.\end{aligned}$$

Exercise 3.2 Find the angle θ between vectors **A** and **B** of Example 3-1 using the cross product between them.

Solution:

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}$$

$$\begin{aligned}
\sin\theta_{AB} &= \frac{|\mathbf{A} \times \mathbf{B}|}{AB} \\
&= \frac{|(\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3) \times (-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}})|}{\sqrt{22}\sqrt{27}} \\
&= \frac{|-\hat{\mathbf{z}}10 + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}3 - \hat{\mathbf{x}}3 - \hat{\mathbf{y}}3 + \hat{\mathbf{x}}15|}{\sqrt{22}\sqrt{27}} \\
&= \frac{\hat{\mathbf{x}}12 - \hat{\mathbf{y}} - \hat{\mathbf{z}}7|}{\sqrt{22}\sqrt{27}} = \frac{\sqrt{144+1+49}}{\sqrt{22}\sqrt{27}} = 0.57
\end{aligned}$$

$$\theta_{AB} = \sin^{-1}(0.57) = 34.9^\circ \text{ or } 145.1^\circ.$$

Exercise 3.3 Find the angle that vector **B** of Example 3-1 makes with the z-axis.

Solution:

$$\mathbf{B} \cdot \hat{\mathbf{z}} = B \cos\theta$$

$$(-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = \sqrt{27} \cos\theta$$

$$\cos\theta = \frac{-1}{\sqrt{27}}$$

$$\theta = 101.1^\circ.$$

Exercise 3.4 Vectors **A** and **B** lie in the y-z plane and both have the same magnitude of 2 (Fig. E3.4). Determine (a) $\mathbf{A}\cdot\mathbf{B}$ and (b) $\mathbf{A}\times\mathbf{B}$.

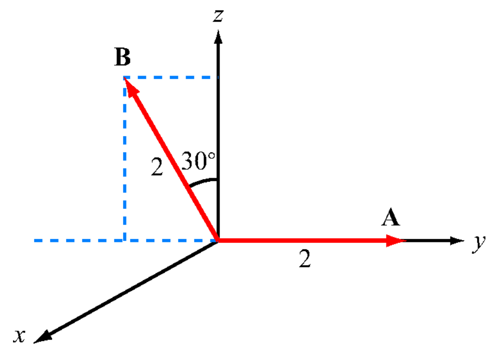


Figure E3.4

Solution:

(a)

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= AB\cos(90^\circ+30^\circ)\\ &= 2\times 2\times\cos 120^\circ\\ &= -2.\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{y}}\,2\\ \mathbf{B} &= -\hat{\mathbf{y}}\,2\cos 60^\circ+\hat{\mathbf{z}}\,2\cos 30^\circ\\ &= -\hat{\mathbf{y}}\,1+\hat{\mathbf{z}}\,1.73\\ \mathbf{A}\times\mathbf{B} &= \hat{\mathbf{y}}\,2\times(-\hat{\mathbf{y}}\,1+\hat{\mathbf{z}}\,1.73)\\ &= \hat{\mathbf{x}}\,3.46.\end{aligned}$$

Exercise 3.5 If $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{C}$, does it follow that $\mathbf{B} = \mathbf{C}$?

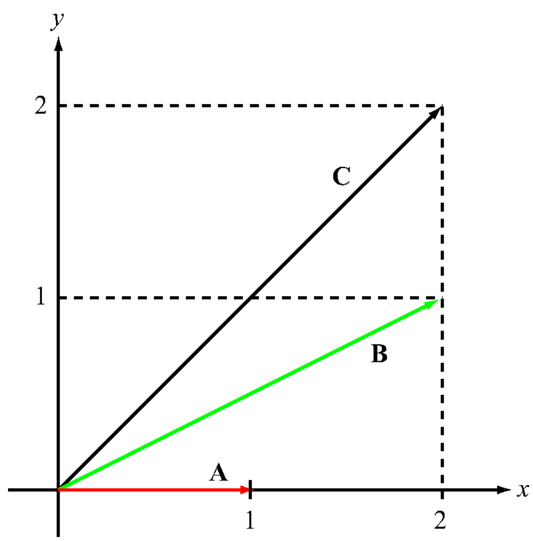
Solution: The answer is No, which can be demonstrated through the following example. Let

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{x}}\,1,\\ \mathbf{B} &= \hat{\mathbf{x}}\,2+\hat{\mathbf{y}}\,1,\\ \mathbf{C} &= \hat{\mathbf{x}}\,2+\hat{\mathbf{y}}\,2.\end{aligned}$$

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= 2,\\ \mathbf{A}\cdot\mathbf{C} &= 2,\end{aligned}$$

but

$$\mathbf{B} \neq \mathbf{C}.$$



Exercise 3.6 A circular cylinder of radius $r = 5$ cm is concentric with the z-axis and extends between $z = -3$ cm and $z = 3$ cm. Use Eq. (3.44) to find the cylinder's volume.

Solution:

$$\begin{aligned}
d\mathcal{V} &= r\,dr\,d\phi\,dz \\
\mathcal{V} &= \int_{r=0}^{5\text{ cm}} \int_{\phi=0}^{2\pi} \int_{z=-3\text{ cm}}^{3\text{ cm}} r\,dr\,d\phi\,dz \\
&= \left.\frac{r^2}{2}\right|_0^{5\text{ cm}} \times \phi\big|_0^{2\pi} \times z\big|_{-3\text{ cm}}^{3\text{ cm}} \\
&= \frac{25}{2} \times 2\pi \times 6 = 471.2\text{ cm}^3.
\end{aligned}$$

Exercise 3.7 Point $P = (2\sqrt{3}, \pi/3, -2)$ is given in cylindrical coordinates. Express P in spherical coordinates.

Solution:

$$R = \sqrt[+]{r^2 + z^2} = \sqrt[+]{(2\sqrt{3})^2 + (-2)^2} = 4$$

$$\phi = \frac{\pi}{3} \qquad \text{(unchanged)}$$

$$\theta = \tan^{-1}\left(\frac{r}{z}\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) = -60^\circ = -\frac{\pi}{3} \text{ or } \frac{2\pi}{3}\,.$$

Exercise 3.8 Transform vector

$$\mathbf{A} = \hat{\mathbf{x}}(x+y) + \hat{\mathbf{y}}(y-x) + \hat{\mathbf{z}}z$$

from Cartesian to cylindrical coordinates.

Solution:

$$\begin{aligned}
\mathbf{A} &= \hat{\mathbf{x}}(x+y) + \hat{\mathbf{y}}(y-x) + \hat{\mathbf{z}}z \\
&= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(r\cos\phi + r\sin\phi) \\
&\quad + (\hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)(r\sin\phi - r\cos\phi) + \hat{\mathbf{z}}z \\
&= \hat{\mathbf{r}}(\cos^2\phi + \cos\phi\sin\phi + \sin^2\phi - \cos\phi\sin\phi)r \\
&\quad + \hat{\boldsymbol{\phi}}(-\sin\phi\cos\phi - \sin^2\phi + \sin\phi\cos\phi - \cos^2\phi)r + \hat{\mathbf{z}}z \\
&= \hat{\mathbf{r}}r - \hat{\boldsymbol{\phi}}r + \hat{\mathbf{z}}z.
\end{aligned}$$

Exercise 3.9 Given $V = x^2y + xy^2 + xz^2$, (a) find the gradient of V, and (b) evaluate it at $(1, -1, 2)$.

Solution:

$$V = x^2y + xy^2 + xz^2$$

(a)

$$\begin{aligned}
\nabla V &= \hat{\mathbf{x}}\,\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\,\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\,\frac{\partial V}{\partial z} \\
&= \hat{\mathbf{x}}\,(2xy + y^2 + z^2) + \hat{\mathbf{y}}\,(x^2 + 2xy) + \hat{\mathbf{z}}\,2xz.
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla V\big|_{(1,-1,2)} &= \hat{\mathbf{x}}\,(-2 + 1 + 4) + \hat{\mathbf{y}}\,(1 - 2) + \hat{\mathbf{z}}\,4 \\
&= \hat{\mathbf{x}}\,3 - \hat{\mathbf{y}} + \hat{\mathbf{z}}\,4.
\end{aligned}$$

Exercise 3.10 Find the directional derivative of $V = rz^2 \cos 2\phi$ along the direction $\mathbf{A} = \hat{\mathbf{r}}2 - \hat{\mathbf{z}}$ and evaluate it at $(1, \pi/2, 2)$.

Solution:

$$\begin{aligned}
V &= rz^2 \cos 2\phi \\
\nabla V &= \hat{\mathbf{r}}\,\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\,\frac{1}{r}\,\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\,\frac{\partial V}{\partial z} \\
&= \hat{\mathbf{r}}\,z^2 \cos 2\phi - \hat{\boldsymbol{\phi}}\,\frac{2r}{r} z^2 \sin 2\phi + \hat{\mathbf{z}}\,2rz \cos 2\phi
\end{aligned}$$

$$\begin{aligned}
\frac{dV}{dl} &= \nabla V \cdot \hat{\mathbf{a}}_l \\
&= \nabla V \cdot \frac{\mathbf{A}}{A} \\
&= (\hat{\mathbf{r}}\,z^2 \cos 2\phi - \hat{\boldsymbol{\phi}}\,2z^2 \sin 2\phi + \hat{\mathbf{z}}\,2rz \cos 2\phi) \cdot \frac{\hat{\mathbf{r}}\,2 - \hat{\mathbf{z}}}{\sqrt{5}} \\
&= \frac{2z^2 \cos 2\phi - 2rz \cos 2\phi}{\sqrt{5}}
\end{aligned}$$

$$\begin{aligned}
\left.\frac{dV}{dl}\right|_{(1,\pi/2,2)} &= \frac{2 \times 4 \cos \pi - 2 \times 2 \cos \pi}{\sqrt{5}} \\
&= -4/\sqrt{5}.
\end{aligned}$$

Exercise 3.11 The power density radiated by a star [Fig. E3.11(a)] decreases radially as $S(R) = S_0/R^2$, where R is the radial distance from the star and S_0 is a constant. Recalling that the gradient of a scalar function denotes the maximum rate of change of that function per unit distance and the direction of the gradient is along the direction of maximum increase, generate an arrow representation of ∇S.

(a)

(b)

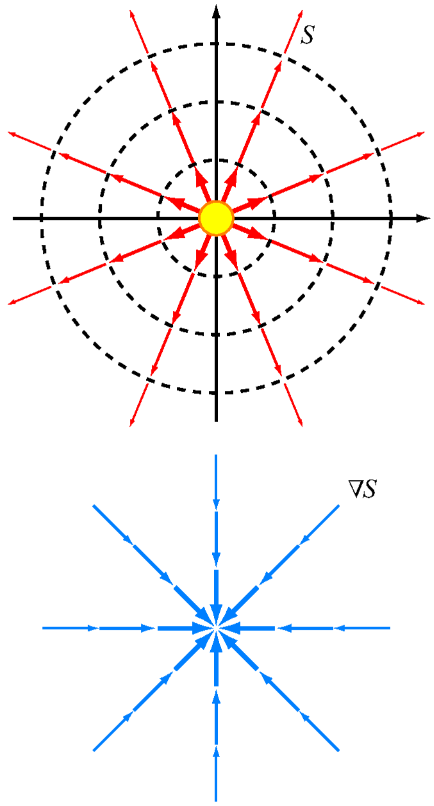


Figure E3.11

Solution:

$$
\begin{aligned}
S(R) &= \frac{S_0}{R^2}\,. \\
\nabla S &= \left(\hat{\mathbf{R}}\,\frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}}\,\frac{1}{R}\,\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{R\sin\theta}\,\frac{\partial}{\partial \phi} \right) \frac{S_0}{R^2} \\
&= -\hat{\mathbf{R}}\,2\,\frac{S_0}{R^3}\,.
\end{aligned}
$$

Exercise 3.12 The graph in Fig. E3.12(a) depicts a gentle change in atmospheric temperature from T_1 over the sea to T_2 over land. The temperature profile is described by the function

$$T(x) = T_1 + (T_2 - T_1)/(e^{-x} + 1),$$

where x is measured in kilometers and $x = 0$ is the sea-land boundary. (a) In which direction does ∇T point and (b) at what value of x is it a maximum?

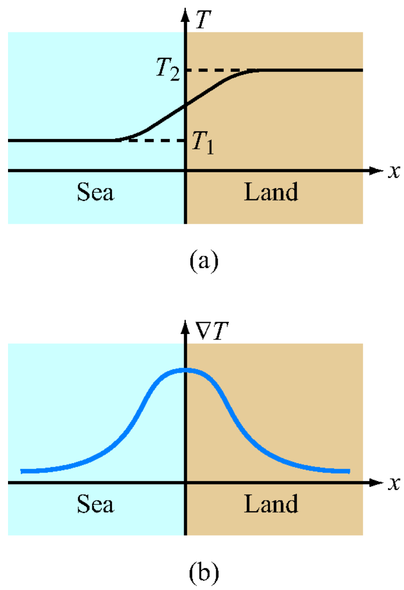


Figure E3.12

Solution:

$$\begin{aligned}
T(x) &= T_1 + \frac{T_2 - T_1}{e^{-x} + 1} \\
\nabla T &= \hat{\mathbf{x}}\, \frac{\partial T}{\partial x} \\
&= \hat{\mathbf{x}}\, \frac{\partial}{\partial x} \left(T_1 + \frac{T_2 - T_1}{e^{-x} + 1} \right) \\
&= \hat{\mathbf{x}}(T_2 - T_1)\, \frac{\partial}{\partial x} (e^{-x} + 1)^{-1} \\
&= \hat{\mathbf{x}}(T_2 - T_1) e^{-x} (e^{-x} + 1)^{-2} \\
&= \hat{\mathbf{x}}\, \frac{e^{-x}(T_2 - T_1)}{(e^{-x} + 1)^2}\ .
\end{aligned}$$

Exercise 3.13 Given $\mathbf{A} = e^{-2y}(\hat{\mathbf{x}}\sin 2x + \hat{\mathbf{y}}\cos 2x)$, find $\nabla \cdot \mathbf{A}$.

Solution:

$$
\begin{aligned}
\mathbf{A} &= e^{-2y}(\hat{\mathbf{x}}\,\sin 2x + \hat{\mathbf{y}}\,\cos 2x) \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
&= \frac{\partial}{\partial x}(e^{-2y}\sin 2x) + \frac{\partial}{\partial y}(e^{-2y}\cos 2x) \\
&= 2e^{-2y}\cos 2x - 2e^{-2y}\cos 2x = 0.
\end{aligned}
$$

Exercise 3.14 Given $\mathbf{A} = \hat{\mathbf{r}}\, r\cos\phi + \hat{\boldsymbol{\phi}}\, r\sin\phi + \hat{\mathbf{z}}\, 3z$, find $\nabla\cdot\mathbf{A}$ at $(2,0,3)$.

Solution:

$$\begin{aligned}
\mathbf{A} &= \hat{\mathbf{r}}\, r\cos\phi + \hat{\boldsymbol{\phi}}\, r\sin\phi + \hat{\mathbf{z}}\, 3z \\
\nabla\cdot\mathbf{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \\
&= \frac{1}{r}\frac{\partial}{\partial r}(r^2\cos\phi) + \frac{1}{r}\frac{\partial}{\partial\phi}(r\sin\phi) + \frac{\partial}{\partial z}(3z) \\
&= 2\cos\phi + \cos\phi + 3
\end{aligned}$$

$$\nabla\cdot\mathbf{A}\big|_{(2,0,3)} = 2+1+3 = 6.$$

Exercise 3.15 If $\mathbf{E} = \hat{\mathbf{R}}AR$ in spherical coordinates, calculate the flux of $\mathbf{E}$ through a spherical surface of radius a, centered at the origin.

Solution:

$$\begin{aligned}
\mathbf{E} &= \hat{\mathbf{R}}AR \\
\oint_S \mathbf{E} \cdot d\mathbf{s} &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \hat{\mathbf{R}}AR \cdot \hat{\mathbf{R}}R^2 \sin\theta \; d\theta \; d\phi \Big|_{R=a} \\
&= \left[-2\pi AR^3 \cos\theta \Big|_0^{\pi} \right]_{R=a} \\
&= 4\pi Aa^3.
\end{aligned}$$

Exercise 3.16 Verify the divergence theorem by calculating the volume integral of the divergence of the field **E** of Exercise 3.11 over the volume bounded by the surface of radius a.

Solution:

$$\text{Divergence Theorem:} \qquad \int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

From Exercise 3.11,

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = 4\pi A a^3.$$

For the left side of Divergence Theorem, with $\mathbf{E} = \hat{\mathbf{R}} AR$,

$$\nabla \cdot \mathbf{E} = \frac{1}{R^2} \frac{\partial}{\partial R}(R^2 E_R) = \frac{1}{R^2} \frac{\partial}{\partial R}(AR^3) = 3A$$

$$\begin{aligned}
\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\mathcal{V} &= \int_0^a \int_0^\pi \int_0^{2\pi} 3A \cdot R^2 \sin\theta \; dR\, d\theta\, d\phi \\
&= \left.\frac{3AR^3}{3}\right|_0^a \times (-\cos\theta|_0^\pi) \times \pi|_0^{2\pi} \\
&= 4\pi A a^3.
\end{aligned}$$

Hence, Divergence Theorem is verified.

Exercise 3.17 The arrow representation in Fig. E3.17 represents the vector field $\mathbf{A} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$. At a given point in space, $\mathbf{A}$ has a positive divergence $\nabla \cdot \mathbf{A}$ if the net flux flowing outward through the surface of an imaginary infinitesimal volume centered at that point is positive, $\nabla \cdot \mathbf{A}$ is negative if the net flux is into the volume, and $\nabla \cdot \mathbf{A} = 0$ if the same amount of flux enters into the volume as leaves it. Determine $\nabla \cdot \mathbf{A}$ everywhere in the x–y plane.

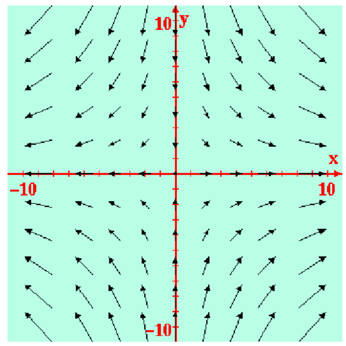


Figure E3.17

Solution:

$$\begin{aligned}
\mathbf{A} &= \hat{\mathbf{x}}x - \hat{\mathbf{y}}y \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
&= \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} \\
&= 1 - 1 = 0.
\end{aligned}$$

Exercise 3.18 Find $\nabla \times \mathbf{A}$ at $(2,0,3)$ in cylindrical coordinates for the vector field

$$\mathbf{A} = \hat{\mathbf{r}} 10 e^{-2r} \cos\phi + \hat{\mathbf{z}} 10 \sin\phi .$$

Solution:

$$\begin{aligned}
\mathbf{A} &= \hat{\mathbf{r}} 10 e^{-2r} \cos\phi + \hat{\mathbf{z}} 10 \sin\phi \\
\nabla \times \mathbf{A} &= \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{\mathbf{z}} \frac{1}{r} \left(\frac{\partial}{\partial r} r A_\phi - \frac{\partial A_r}{\partial \phi} \right) \\
&= \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial}{\partial \phi} (10 \sin\phi) \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial}{\partial z} (10 e^{-2r} \cos\phi) - \frac{\partial}{\partial r} (10 \sin\phi) \right) \\
&\quad + \hat{\mathbf{z}} \frac{1}{r} \frac{\partial}{\partial \phi} (-10 e^{-2r} \cos\phi) \\
&= \hat{\mathbf{r}} \frac{10 \cos\phi}{r} + \hat{\mathbf{z}} \frac{10 e^{-2r}}{r} \sin\phi
\end{aligned}$$

$$\nabla \times \mathbf{A} \big|_{(2,0,3)} = \hat{\mathbf{r}} 5.$$

Exercise 3.19 Find $\nabla \times \mathbf{A}$ at $(3, \pi/6, 0)$ in spherical coordinates for the vector field $\mathbf{A} = \hat{\boldsymbol{\theta}}\, 12\sin\theta$.

Solution:

$$
\begin{aligned}
\mathbf{A} &= \hat{\boldsymbol{\theta}}\, 12\sin\theta \\
\nabla \times \mathbf{A} &= \hat{\boldsymbol{\phi}}\, \frac{1}{R}\frac{\partial}{\partial R}(RA_\theta) \\
&= \hat{\boldsymbol{\phi}}\, \frac{1}{R}\frac{\partial}{\partial R}(12R\sin\theta) \\
&= \hat{\boldsymbol{\phi}}\, \frac{12\sin\theta}{R}\ .
\end{aligned}
$$

$$
\nabla \times \mathbf{A}\big|_{(3,\pi/6,0)} = \hat{\boldsymbol{\phi}}\, 4\sin 30^\circ = \hat{\boldsymbol{\phi}}\, 2.
$$

Chapter 4 Exercise Solutions

Exercise 4.1 A square plate in the x–y plane is situated in the space defined by -3 m $\leq x \leq 3$ m and -3 m $\leq y \leq 3$ m. Find the total charge on the plate if the surface charge density is given by $\rho_s = 4y^2$ (μC/m^2).

Solution:

$$\begin{aligned}
\rho_s &= 4y^2 \\
Q &= \int_S \rho_s \, ds \\
&= \int_{-3}^{3} \int_{-3}^{3} 4y^2 \, dx \, dy \\
&= \left. \left. \frac{4y^3 x}{3} \right|_{-3}^{3} \right|_{-3}^{3} = 432 \ \mu\text{C} = 0.432 \quad \text{(mC)}.
\end{aligned}$$

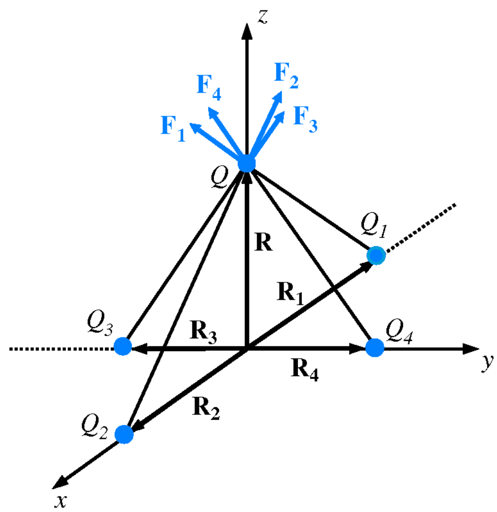
Exercise 4.2 A spherical shell centered at the origin extends between $R = 2$ cm and $R = 3$ cm. If the volume charge density is given by $\rho_v = 3R \times 10^{-4}$ (C/m^3), find the total charge contained in the shell.

Solution:

$$\begin{aligned}
\rho_v &= 3R \times 10^{-4} \\
Q &= \int_{\mathscr{V}} \rho_v \, d\mathscr{V} \\
&= \int_{R=2\text{ cm}}^{3\text{ cm}} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 3R \times 10^{-4} \cdot R^2 \sin\theta \, dR \, d\theta \, d\phi \\
&= \frac{3R^4}{4} \times 10^{-4} \Big|_{2\text{ cm}}^{3\text{ cm}} \times 2 \times 2\pi \\
&= 3\pi \times 10^{-4} \left[(3 \times 10^{-2})^4 - (2 \times 10^{-2})^4 \right] = 0.61 \quad \text{(nC)}.
\end{aligned}$$

Exercise 4.3 Four charges of 10 μC each are located in free space at points with Cartesian coordinates $(-3,0,0)$, $(3,0,0)$, $(0,-3,0)$, and $(0,3,0)$. Find the force on a 20-μC charge located at $(0,0,4)$. All distances are in meters.

Solution:



$$\mathbf{R}_1 = -\hat{\mathbf{x}}3$$
$$\mathbf{R}_2 = \hat{\mathbf{x}}3$$
$$\mathbf{R}_3 = -\hat{\mathbf{y}}3$$
$$\mathbf{R}_4 = \hat{\mathbf{y}}3$$
$$\mathbf{R} = \hat{\mathbf{z}}4$$
$$\mathbf{F}_1 = \frac{QQ_1}{4\pi\varepsilon_0}\frac{\mathbf{R}-\mathbf{R}_1}{|\mathbf{R}-\mathbf{R}_1|^3} = \frac{QQ_1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{z}}4+\hat{\mathbf{x}}3}{125} = \frac{QQ_1}{500\pi\varepsilon_0}(\hat{\mathbf{z}}4+\hat{\mathbf{x}}3)$$
$$\mathbf{F}_2 = \frac{QQ_2}{4\pi\varepsilon_0}\frac{\mathbf{R}-\mathbf{R}_2}{|\mathbf{R}-\mathbf{R}_2|^3} = \frac{QQ_2}{4\pi\varepsilon_0}\frac{\hat{\mathbf{z}}4-\hat{\mathbf{x}}3}{125} = \frac{QQ_2}{500\pi\varepsilon_0}(\hat{\mathbf{z}}4-\hat{\mathbf{x}}3)$$
$$\mathbf{F}_3 = \frac{QQ_3}{4\pi\varepsilon_0}\frac{\mathbf{R}-\mathbf{R}_3}{|\mathbf{R}-\mathbf{R}_3|^3} = \frac{QQ_3}{4\pi\varepsilon_0}\frac{\hat{\mathbf{z}}4+\hat{\mathbf{y}}3}{125} = \frac{QQ_3}{500\pi\varepsilon_0}(\hat{\mathbf{z}}4+\hat{\mathbf{y}}3)$$
$$\mathbf{F}_4 = \frac{QQ_4}{4\pi\varepsilon_0}\frac{\mathbf{R}-\mathbf{R}_4}{|\mathbf{R}-\mathbf{R}_4|^3} = \frac{QQ_4}{4\pi\varepsilon_0}\frac{\hat{\mathbf{z}}4-\hat{\mathbf{y}}3}{125} = \frac{QQ_4}{500\pi\varepsilon_0}(\hat{\mathbf{z}}4-\hat{\mathbf{y}}3)$$

$$\mathbf{F} = \mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3+\mathbf{F}_4$$
$$= \frac{200\times10^{-12}}{500\pi\varepsilon_0}(\hat{\mathbf{z}}16) = \hat{\mathbf{z}}\frac{32\times10^{-12}}{5\pi\times8.85\times10^{-12}} = \hat{\mathbf{z}}0.23 \quad \text{(N)}.$$

Exercise 4.4 Two identical charges are located on the x-axis at $x = 3$ and $x = 7$. At what point in space is the net electric field zero?

Solution: Since both charges are on the x-axis, the point at which the fields due to the two charges can cancel has to lie on the x-axis also. Intuitively, since the two charges are identical, that point is midway between them at $(5, 0, 0)$.

Exercise 4.5 In a hydrogen atom the electron and proton are separated by an average distance of 5.3×10^{-11} m. Find the magnitude of the electrical force F_e between the two particles, and compare it with the gravitational force F_g between them.

Solution:

$$F_\mathrm{e} = \frac{q_\mathrm{e}q_\mathrm{p}}{4\pi\varepsilon_0 R^2} = \frac{(1.6\times10^{-19})^2}{4\pi\times8.85\times10^{-12}(5.3\times10^{-11})^2} = 8.2\times10^{-8}\text{ N}.$$

$$F_\mathrm{g} = \frac{Gm_\mathrm{e}m_\mathrm{p}}{R^2} = \frac{6.67\times10^{-11}\times9.11\times10^{-31}\times1.67\times10^{-27}}{(5.3\times10^{-11})^2} = 3.6\times10^{-47}\text{ N}.$$

Exercise 4.6 An infinite sheet of charge with uniform surface charge density ρ_s is located at $z = 0$ (x–y plane), and another infinite sheet with density $-\rho_s$ is located at $z = 2$ m, both in free space. Determine **E** in all regions.

Solution: Per Eq. (4.25), for the sheet at $z = 0$,

$$\mathbf{E}_1 = \begin{cases} \hat{\mathbf{z}}\,\dfrac{\rho_s}{2\varepsilon_0}\,, & \text{for } z > 0, \\ -\hat{\mathbf{z}}\,\dfrac{\rho_s}{2\varepsilon_0}\,, & \text{for } z < 0. \end{cases}$$

Similarly, for the sheet at $z = 2$ m with charge density $-\rho_s$,

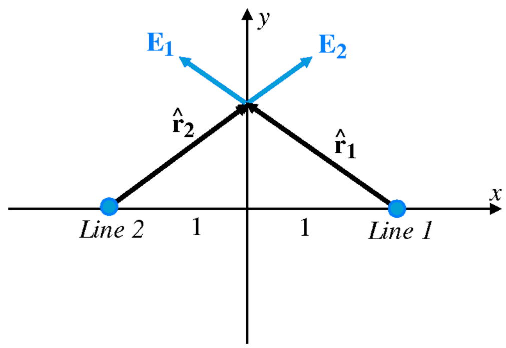
$$\mathbf{E}_2 = \begin{cases} -\hat{\mathbf{z}}\,\dfrac{\rho_s}{2\varepsilon_0}\,, & \text{for } z > 2 \text{ m}, \\ \hat{\mathbf{z}}\,\dfrac{\rho_s}{2\varepsilon_0}\,, & \text{for } z < 2 \text{ m}. \end{cases}$$

Hence,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} 0, & \text{for } z < 0, \\ \hat{\mathbf{z}}\,\dfrac{\rho_s}{\varepsilon_0}\,, & \text{for } 0 < z < 2 \text{ m}, \\ 0, & \text{for } z > 2 \text{ m}. \end{cases}$$

Exercise 4.7 Two infinite lines of charge, each carrying a charge density ρ_l, are parallel to the z-axis and located at $x = 1$ and $x = -1$. Determine **E** at an arbitrary point in free space along the y-axis.

Solution:



The distance between either line of charge and a point at y on the y-axis is $r = (1+y^2)^{1/2}$.

For line 1,

$$\hat{\mathbf{r}}_1 = \frac{\mathbf{r}_1}{r} = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}\,y}{(1+y^2)^{1/2}} \,.$$

For line 2,

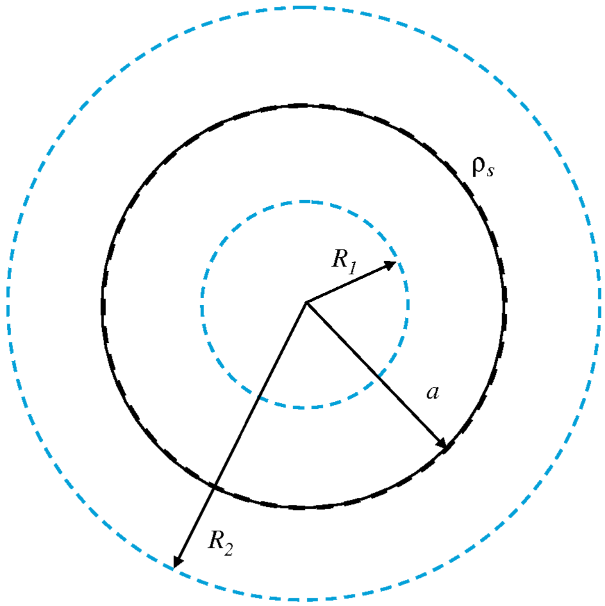
$$\hat{\mathbf{r}}_2 = \frac{\mathbf{r}_2}{r} = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}\,y}{(1+y^2)^{1/2}} \,.$$

Using Eq. (4.33),

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 \\ &= \frac{\hat{\mathbf{r}}_1 \rho_l}{2\pi\varepsilon_0 r} + \frac{\hat{\mathbf{r}}_2 \rho_l}{2\pi\varepsilon_0 r} \\ &= \frac{(-\hat{\mathbf{x}} + \hat{\mathbf{y}}\,y)\rho_l}{2\pi\varepsilon_0(1+y^2)} + \frac{(\hat{\mathbf{x}} + \hat{\mathbf{y}}\,y)\rho_l}{2\pi\varepsilon_0(1+y^2)} = \frac{\hat{\mathbf{y}}\,\rho_l y}{\pi\varepsilon_0(y^2+1)} \,. \end{aligned}$$

Exercise 4.8 A thin spherical shell of radius a carries a uniform surface charge density ρ_s. Use Gauss's law to determine $\mathbf{E}$.

Solution:



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Symmetry suggests that $\mathbf{D}$ is radial in direction. Hence,

$$\mathbf{D} = \hat{\mathbf{R}} D_R$$
$$d\mathbf{s} = \hat{\mathbf{R}}\, ds$$
$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \oint_S D_R\, ds = D_R(4\pi R^2) = Q$$
$$D_R = \frac{Q}{4\pi R^2}$$

- For a Gaussian surface of radius $R_1 < a$, no charge is enclosed. Hence, $Q = 0$, in which case $E = 0$.
- For a Gaussian surface of radius $R_2 > a$,

  $$Q = \rho_s(4\pi a^2)$$

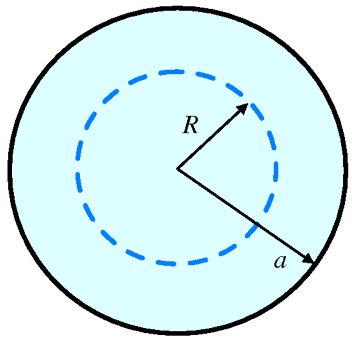
  and

  $$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{\hat{\mathbf{R}}}{\varepsilon} D_r = \frac{\hat{\mathbf{R}} Q}{4\pi\varepsilon R_2^2} = \hat{\mathbf{R}} \frac{4\pi \rho_s a^2}{4\pi\varepsilon R_2^2} = \hat{\mathbf{R}} \frac{\rho_s a^2}{\varepsilon R_2^2}\,.$$

Exercise 4.9 A spherical volume of radius a contains a uniform volume charge density ρ_v. Use Gauss's law to determine $\mathbf{D}$ for (a) $R \le a$ and (b) $R \ge a$.

Solution:

(a)



$R < a$

For $R \le a$,

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = \oint_S D_r\, ds = D_r(4\pi R^2)$$

Q within a sphere of radius R is

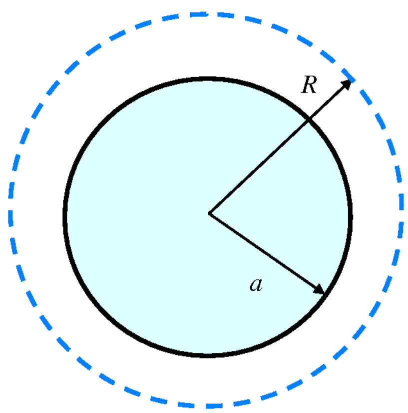
$$Q = \frac{4}{3}\pi R^3 \rho_\mathrm{v}$$

Hence,

$$4\pi R^2 D_R = \frac{4}{3}\pi R^3 \rho_\mathrm{v}$$

$$D_r = \frac{\rho_\mathrm{v} R}{3}, \qquad \mathbf{D} = \hat{\mathbf{R}} D_r = \hat{\mathbf{R}}\frac{\rho_\mathrm{v} R}{3}, \qquad R \le a.$$

(b)



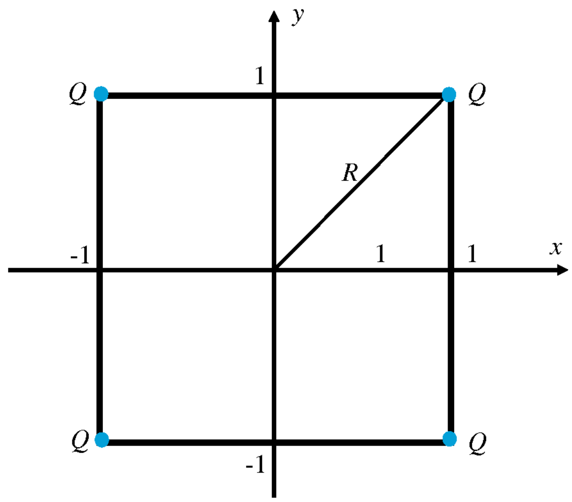
$R > a$

For $R \geq a$, total charge in sphere is

$$Q = \frac{4}{3}\pi a^3 \rho_{\rm v},$$
$$4\pi R^2 D_R = \frac{4}{3}\pi a^3 \rho_{\rm v},$$
$$\mathbf{D} = \hat{\mathbf{R}} D_r = \hat{\mathbf{R}} \frac{\rho_{\rm v} a^3}{3R^2}\ , \qquad R \geq a.$$

Exercise 4.10 Determine the electric potential at the origin due to four 20-μC charges residing in free space at the corners of a 2 m $\times$ 2 m square centered about the origin in the x–y plane.

Solution:



For four identical charges all equidistant from the origin:

$$V = \frac{4Q}{4\pi\varepsilon_0 R}, \qquad R = \sqrt{2} \quad \text{(m)}$$

$$= \frac{4 \times 20 \times 10^{-6}}{4\pi\varepsilon_0\sqrt{2}} = \frac{\sqrt{2} \times 10^{-5}}{\pi\varepsilon_0} \quad \text{(V)}.$$

Exercise 4.11 A spherical shell of radius R has a uniform surface charge density ρ_s. Determine the electric potential at the center of the shell.

Solution: Application of (4.48b):

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{S'} \frac{\rho_s}{R'}\, ds'$$
$$= \frac{1}{4\pi\varepsilon R} \cdot \rho_s (4\pi R^2)$$
$$= \frac{\rho_s R}{\varepsilon}\,.$$

Exercise 4.12 Determine the density of free electrons in aluminum, given that its conductivity is 3.5×10^7 (S/m) and its electron mobility is 0.0015 $(\mathrm{m}^2/\mathrm{V}\cdot\mathrm{s})$.

Solution:

$$\sigma = N_\mathrm{e}\mu_\mathrm{e}e$$

$$N_\mathrm{e} = \frac{\sigma}{\mu_\mathrm{e}e} = \frac{3.5 \times 10^7}{0.0015 \times 1.6 \times 10^{-19}}$$

$$= 1.46 \times 10^{29} \text{ electrons/m}^3.$$

Exercise 4.13 The current flowing through a 100-m-long conducting wire of uniform cross section has a density of 3×10^5 (A/m^2). Find the voltage drop across the length of the wire if the wire material has a conductivity of 2×10^7 (S/m).

Solution:

$$
\begin{aligned}
J &= \sigma E \\
E &= \frac{J}{\sigma} \\
V &= El \\
&= \frac{Jl}{\sigma} \quad \text{(where } l = \text{length of wire)} \\
&= \frac{3 \times 10^5 \times 100}{2 \times 10^7} = 1.5 \quad \text{(V)}.
\end{aligned}
$$

Exercise 4.14 A 50-m-long copper wire has a circular cross section with radius $r = 2$ cm. Given that the conductivity of copper is 5.8×10^7 (S/m), determine (a) the resistance R of the wire and (b) the power dissipated in the wire if the voltage across its length is 1.5 (mV).

Solution:

(a)

$$R = \frac{l}{\sigma A} = \frac{50}{5.8 \times 10^7 \times \pi (0.02)^2}$$
$$= 6.9 \times 10^{-4}\ \Omega.$$

(b)

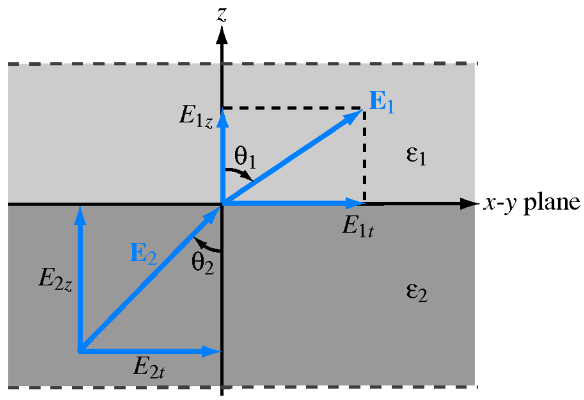
$$P = \frac{V^2}{R} = \frac{(1.5 \times 10^{-3})^2}{6.9 \times 10^{-4}} = 3.3 \quad \text{(mW)}.$$

Exercise 4.15 Repeat part (b) of Exercise 4.14 by applying Eq. (4.80).

Solution:

$$
\begin{aligned}
P &= \int \sigma |E|^2 \, d\mathcal{V} \\
E &= \frac{V}{l} = \frac{1.5 \times 10^{-3}}{50} = 3 \times 10^{-5} \quad \text{(V/m)} \\
P &= \sigma |E|^2 \mathcal{V} \\
&= 5.8 \times 10^7 \times (3 \times 10^{-5})^2 \times 50 \times \pi (0.02)^2 \\
&= 3.3 \quad \text{(mW)}.
\end{aligned}
$$

Exercise 4.16 Find $\mathbf{E}_1$ in Fig. 4-19 if $\mathbf{E}_2 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3$ (V/m), $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 8\varepsilon_0$, and the boundary is charge free.



Solution:

$$\mathbf{E}_2 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3 \quad \text{(V/m)}$$

Given that the x–y plane is the boundary between the two media, the x- and y-components of $\mathbf{E}_2$ are parallel to the boundary, and therefore are the same across the two sides of the boundary. Thus,

$$E_{1x} = E_{2x} = 2$$
$$E_{1y} = E_{2y} = -3.$$

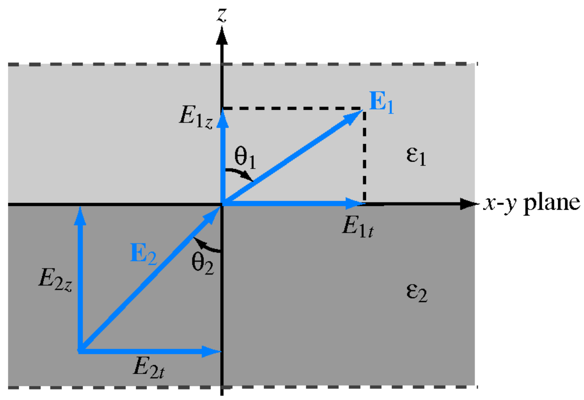
For the z-component,

$$\varepsilon_1 E_{1z} = \varepsilon_2 E_{2z} \qquad (\rho_s = 0)$$
$$E_{1z} = \frac{\varepsilon_2}{\varepsilon_1} E_{2z} = \frac{8\varepsilon_0}{2\varepsilon_0} \cdot 3 = 12.$$

Hence,

$$\mathbf{E}_1 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}12 \quad \text{(V/m)}.$$

Exercise 4.17 Repeat Exercise 4.16 for a boundary with surface charge density $\rho_s = 3.54 \times 10^{-11}$ (C/m^2).



Solution:

$$\mathbf{E}_2 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3$$

From Exercise 4.16,

$$E_{1x} = 2$$
$$E_{1y} = -3.$$

For z-component,

$$\varepsilon_1 E_{1z} - \varepsilon_2 E_{2z} = \rho_s$$
$$E_{1z} = \frac{\varepsilon_2 E_{2z} + \rho_s}{\varepsilon_1}$$
$$= \frac{8\varepsilon_0 \times 3 + 3.54 \times 10^{-11}}{2\varepsilon_0}$$
$$= 12 + \frac{3.54 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}}$$
$$= 12 + 2 = 14 \quad \text{(V/m)}.$$

Hence,

$$\mathbf{E}_1 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}14 \quad \text{(V/m)}.$$

Exercise 4.18 The radii of the inner and outer conductors of a coaxial cable are 2 cm and 5 cm, respectively, and the insulating material between them has a relative permittivity of 4. The charge density on the outer conductor is $\rho_l = 10^{-4}$ (C/m). Use the expression for $\mathbf{E}$ derived in Example 4-12 to calculate the total energy stored in a 20-cm length of the cable.

Solution:

$$\begin{aligned}
E &= \frac{\rho_l}{2\pi\varepsilon r} \\
W_\mathrm{e} &= \frac{1}{2}\int_{\mathscr{V}} \varepsilon E^2 \, d\mathscr{V} \\
&= \frac{1}{2}\varepsilon l \int_{r=2\text{ cm}}^{5\text{ cm}} E^2 (2\pi r \, dr) \\
&= \pi\varepsilon l \int_{2\text{ cm}}^{5\text{ cm}} \left(\frac{\rho_l}{2\pi\varepsilon r}\right)^2 r \, dr \\
&= \frac{\rho_l^2 l}{4\pi\varepsilon} \int_{2\text{ cm}}^{5\text{ cm}} \frac{dr}{r} \\
&= \frac{\rho_l^2 l}{4\pi\varepsilon} \ln\left(\frac{5}{2}\right) = 4.1 \quad \text{(J)}.
\end{aligned}$$

Exercise 4.19 Use the result of Example 4-13 to find the surface charge density ρ_s on the surface of the conducting plane.

Solution: According to (4.95),

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_\mathrm{s}.$$

In the present case, E_{2n} (in conductor) $= 0$, $\varepsilon_1 = \varepsilon_0$ and E_{1n} is the z-component of the expression given in Example 4-13, evaluated at $z = 0$, namely

$$E_{1n} = -\frac{2Qd}{4\pi\varepsilon_0} \cdot \frac{1}{(x^2 + y^2 + d^2)^{3/2}} \, .$$

Hence,

$$\rho_\mathrm{s} = \varepsilon_0 E_{1n} = -\frac{Qd}{2\pi(x^2 + y^2 + d^2)^{3/2}} \, .$$

Chapter 5 Exercise Solutions

Exercise 5.1 An electron moving in the positive x-direction perpendicular to a magnetic field experiences a deflection in the negative z-direction. What is the direction of the magnetic field?

Solution: The magnetic force acting on a moving charged particle is

$$\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B}$$

In this case,

$$\begin{aligned} q &= -e \\ \mathbf{u} &= \hat{\mathbf{x}}u \\ \mathbf{F}_{\mathrm{m}} &= -\hat{\mathbf{z}}F_m \\ -\hat{\mathbf{z}}F_{\mathrm{m}} &= -\hat{\mathbf{x}}ue \times \mathbf{B} \end{aligned}$$

For the cross product to apply, **B** has to be in the positive y-direction.

Exercise 5.2 A proton moving with a speed of 2×10^6 m/s through a magnetic field with magnetic flux density of 2.5 T experiences a magnetic force of magnitude 4×10^{-13} N. What is the angle between the magnetic field and the proton's velocity?

Solution:

$$
\begin{aligned}
F &= quB\sin\theta \\
\sin\theta &= \frac{F}{quB} \\
&= \frac{4 \times 10^{-13}}{1.6 \times 10^{-19} \times 2 \times 10^6 \times 2.5} \\
&= 0.5
\end{aligned}
$$

$$
\theta = \sin^{-1} 0.5 = 30^\circ \text{ or } 150^\circ.
$$

Exercise 5.3 A charged particle with velocity $\mathbf{u}$ is moving in a medium containing uniform fields $\mathbf{E} = \hat{\mathbf{x}}E$ and $\mathbf{B} = \hat{\mathbf{y}}B$. What should $\mathbf{u}$ be so that the particle experiences no net force on it?

Solution:

$$\mathbf{F}_{\mathrm{e}} = q\mathbf{E} = \hat{\mathbf{x}}\,qE$$
$$\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B} = q(\mathbf{u} \times \hat{\mathbf{y}}\,B)$$

For net force to be zero, $\mathbf{F}_{\mathrm{m}}$ has to be along $-\hat{\mathbf{x}}$, which requires $\mathbf{u}$ to be along $+\hat{\mathbf{z}}$. Thus,

$$qE = quB$$
$$u = \frac{E}{B}$$
$$\mathbf{u} = \hat{\mathbf{z}}\,\frac{E}{B}\,.$$

If $\mathbf{u}$ also has a y-component, that component will exercise no force on the particle.

Exercise 5.4 A horizontal wire with a mass per unit length of 0.2 kg/m carries a current of 4 A in the $+x$-direction. If the wire is placed in a uniform magnetic flux density $\mathbf{B}$, what should the direction and minimum magnitude of $\mathbf{B}$ be in order to magnetically lift the wire vertically upward? (Hint: The acceleration due to gravity is $\mathbf{g} = -\hat{\mathbf{z}}9.8$ m/s^2.)

Solution: For a length l,

$$\mathbf{F}_\mathrm{g} = -\hat{\mathbf{z}}0.2l \times 9.8 = -\hat{\mathbf{z}}1.96l \quad \text{(N)}$$
$$\mathbf{F}_\mathrm{m} = \hat{\mathbf{x}}Il \times \mathbf{B}$$

For $\mathbf{F}_\mathrm{m} + \mathbf{F}_\mathrm{g} = 0$, $\mathbf{F}_\mathrm{m}$ has to be along $+\hat{\mathbf{z}}$, which means that $\mathbf{B}$ has to be along $+\hat{\mathbf{y}}$. Hence,

$$1.96l = IlB$$
$$B = \frac{1.96}{I} = 0.49 \text{ (T), and}$$
$$\mathbf{B} = \hat{\mathbf{y}}0.49 \text{ (T).}$$

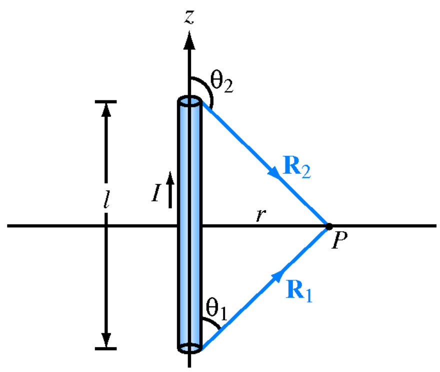
Exercise 5.5 A square coil of 100 turns and 0.5-m-long sides is in a region with a uniform magnetic flux density of 0.2 T. If the maximum magnetic torque exerted on the coil is 4×10^{-2} (N·m), what is the current flowing in the coil?

Solution:

$$T_{\max} = NIAB_0$$

$$I = \frac{T_{\max}}{NAB_0} = \frac{4 \times 10^{-2}}{100 \times (0.5)^2 \times 0.2} = 8 \quad \text{(mA)}.$$

Exercise 5.6 A semiinfinite linear conductor extends between $z = 0$ and $z = \infty$ along the z-axis. If the current I in the conductor flows along the positive z-direction, find $\mathbf{H}$ at a point in the x–y plane at a radial distance r from the conductor.



Solution: From (5.27),

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi r}\,(\cos\theta_1 - \cos\theta_2)$$

For a conductor extending from $z = 0$ to $z = \infty$, $\theta_1 = 0$ and $\theta_2 = \pi$. Hence,

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{I}{4\pi r}\,(1+1) = \hat{\boldsymbol{\phi}}\,\frac{I}{2\pi r} \quad \text{(A/m).}$$

Exercise 5.7 A wire carrying a current of 4 A is formed into a circular loop. If the magnetic field at the center of the loop is 20 A/m, what is the radius of the loop if the loop has (a) only one turn and (b) 10 turns?

Solution:

(a) From (5.35),

$$\mathbf{H} = \hat{\mathbf{z}}\,\frac{I}{2a} \qquad (\text{at } z = 0).$$

$$a = \frac{I}{2H} = \frac{4}{2 \times 20} = 0.1 \quad (\text{m}).$$

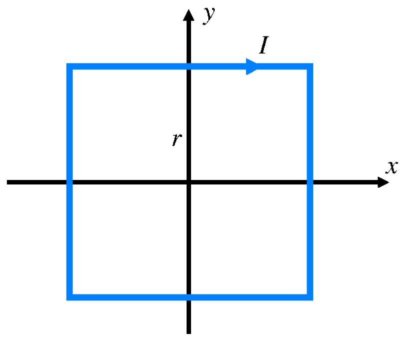
(b)

$$H = \frac{NI}{2a}$$

$$a = \frac{NI}{2H} = 10 \times 0.1 = 1 \quad (\text{m}).$$

Exercise 5.8 A wire is formed into a square loop and placed in the x–y plane with its center at the origin and each of its sides parallel to either the x- or y-axes. Each side is 40 cm in length, and the wire carries a current of 5 A whose direction is clockwise when the loop is viewed from above. Calculate the magnetic field at the center of the loop.

Solution:



The direction of the current will induce a magnetic field along $-\hat{\mathbf{z}}$ (according to the right-hand rule). At the center of the loop, each segment will contribute exactly the same amount. Each of the four contributions can be calculated using (5.29) with $\hat{\boldsymbol{\phi}}$ replaced with $-\hat{\mathbf{z}}$:

$$\mathbf{H}_1 = -\hat{\mathbf{z}}\,\frac{Il}{2\pi r\sqrt{4r^2 + l^2}}\,.$$

In this case $r = l/2$. Hence,

$$\mathbf{H}_1 = -\hat{\mathbf{z}}\,\frac{Il}{2\pi(l/2)\sqrt{l^2 + l^2}} = -\hat{\mathbf{z}}\,\frac{I}{\sqrt{2}\,\pi l}\,.$$

Finally,

$$\begin{aligned} \mathbf{H} = 4\mathbf{H}_1 &= -\hat{\mathbf{z}}\,\frac{4I}{\sqrt{2}\,\pi l} \\ &= -\hat{\mathbf{z}}\,\frac{4 \times 5}{\sqrt{2}\,\pi \times 0.4} = -\hat{\mathbf{z}}\,11.25 \quad \text{(A/m)}. \end{aligned}$$

Exercise 5.9 Current I flows in the inner conductor of a long coaxial cable and returns through the outer conductor. What is the magnetic field in the region outside the coaxial cable and why?

Solution: $\mathbf{H} = 0$ outside the coaxial cable because the net current enclosed by the Ampèrian contour is zero.

Exercise 5.10 The metal niobium becomes a superconductor with zero electrical resistance when it is cooled to below 9 K, but its superconductive behavior ceases when the magnetic flux density at its surface exceeds 0.12 T. Determine the maximum current that a 0.1-mm-diameter niobium wire can carry and remain superconductive.

Solution: From (5.49), the magnetic field at $r \geq a$ from a wire is given by

$$H = \frac{I}{2\pi r}\,, \qquad r \geq a.$$

At the surface of the wire, $r = a$. Hence,

$$\begin{aligned}
B &= \mu_0 H = \frac{\mu_0 I}{2\pi a}\,, \\
I &= \frac{2\pi a B}{\mu_0} \\
&= \frac{2\pi \times 0.05 \times 10^{-3} \times 0.12}{4\pi \times 10^{-7}} = 30 \text{ A}.
\end{aligned}$$

Exercise 5.11 The magnetic vector $\mathbf{M}$ is the vector sum of the magnetic moments of all the atoms contained in a unit volume (1m^3). If a certain type of iron with 8.5×10^{28} atoms/m^3 contributes one electron per atom to align its spin magnetic moment along the direction of the applied field, find (a) the spin magnetic moment of a single electron, given that $m_e = 9.1 \times 10^{-31}$ (kg) and $\hbar = 1.06 \times 10^{-34}$ (J·s), and (b) the magnitude of $\mathbf{M}$.

Solution:

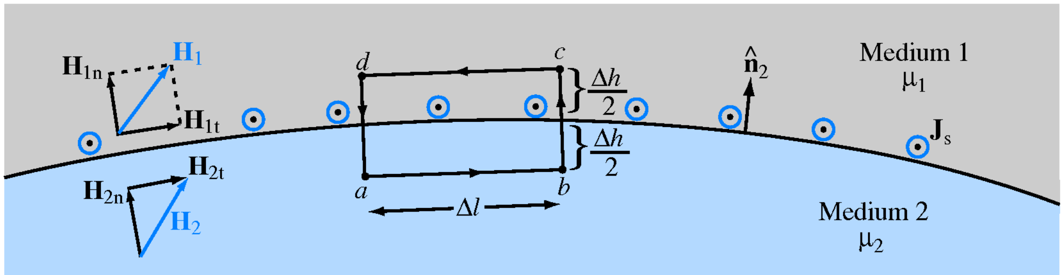
(a)

$$m_s = \frac{e\hbar}{2m_e} = \frac{1.6 \times 10^{-19} \times 1.06 \times 10^{-34}}{2 \times 9.1 \times 10^{-31}} = 9.3 \times 10^{-24} \quad (\text{A}\cdot\text{m}^2).$$

(b)

$$M = nm_s = 8.5 \times 10^{28} \times 9.3 \times 10^{-24} = 7.9 \times 10^5 \quad (\text{A/m}).$$

Exercise 5.12 With reference to Fig. 5-24, determine the angle between $\mathbf{H}_1$ and $\hat{\mathbf{n}}_2 = \hat{\mathbf{z}}$ if $\mathbf{H}_2 = (\hat{\mathbf{x}}3 + \hat{\mathbf{z}}2)$ (A/m), $\mu_{r_1} = 2$, and $\mu_{r_2} = 8$, and $\mathbf{J}_s = 0$.



Solution:

$$\mathbf{H}_2 = \hat{\mathbf{x}}3 + \hat{\mathbf{z}}2$$
$$H_{1x} = H_{2x} = 3$$
$$\mu_1 H_{1z} = \mu_2 H_{2z}$$
$$H_{1z} = \frac{\mu_2}{\mu_1} H_{2z} = \frac{8}{2} \times 2 = 8$$

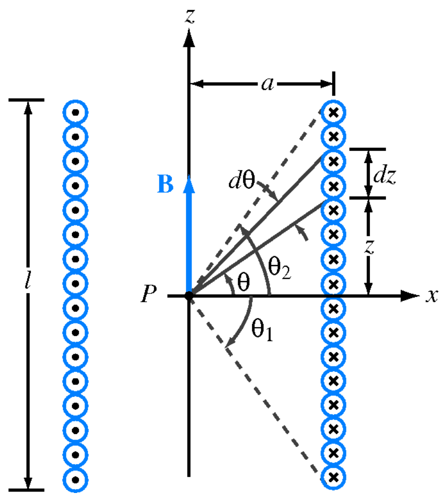
Hence,

$$\mathbf{H}_1 = \hat{\mathbf{x}}3 + \hat{\mathbf{z}}8$$
$$\mathbf{H}_1 \cdot \hat{\mathbf{z}} = H_1 \cos\theta$$

$$\cos\theta = \frac{\mathbf{H}_1 \cdot \hat{\mathbf{z}}}{H_1} = \frac{8}{\sqrt{9+64}} = \frac{8}{\sqrt{73}} = 0.936$$

$$\theta = 20.6^\circ.$$

Exercise 5.13 Use Eq. (5.89) to obtain an expression for $\mathbf{B}$ at a point on the axis of a very long solenoid but situated at its end points. How does $\mathbf{B}$ at the end points compare to $\mathbf{B}$ at the midpoint of the solenoid?



Solution:

$$\mathbf{B} = \hat{\mathbf{z}} \frac{\mu n I}{2} (\sin \theta_2 - \sin \theta_1)$$

For a point at P with $\theta_1 = 0$ and $\theta_2 = 90°$,

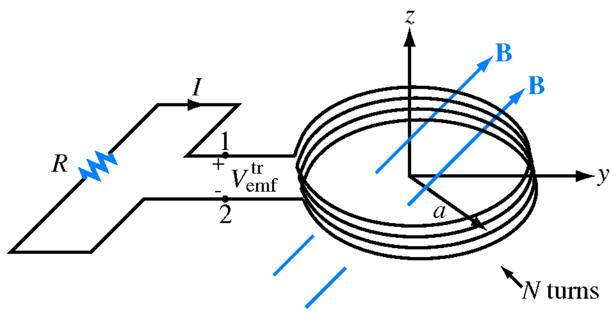
$$\mathbf{B} = \hat{\mathbf{z}} \frac{\mu n I}{2} = \hat{\mathbf{z}} \frac{\mu N I}{2l} ,$$

which is half as large as $\mathbf{B}$ at the midpoint.

Chapter 6 Exercise Solutions

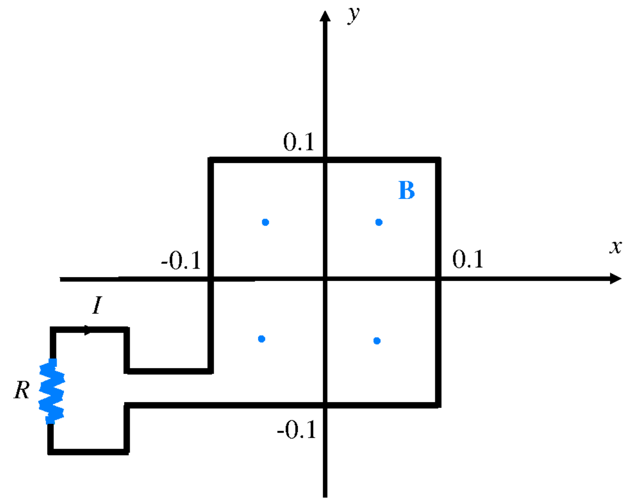
Exercise 6.1 For the loop shown in Fig. 6-3, what is $V_{\text{emf}}^{\text{tr}}$ if $\mathbf{B} = \hat{\mathbf{y}} B_0 \cos \omega t$?

Solution: $V_{\text{emf}}^{\text{tr}} = 0$ because $\mathbf{B}$ is orthogonal to the loop's surface normal $d\mathbf{s}$.



Exercise 6.2 Suppose that the loop of Example 6-1 is replaced with a 10-turn square loop centered at the origin and having 20-cm sides oriented parallel to the x- and y-axes. If $\mathbf{B} = \hat{\mathbf{z}} B_0 x^2 \cos 10^3 t$ and $B_0 = 100$ T, find the current in the circuit.

Solution:



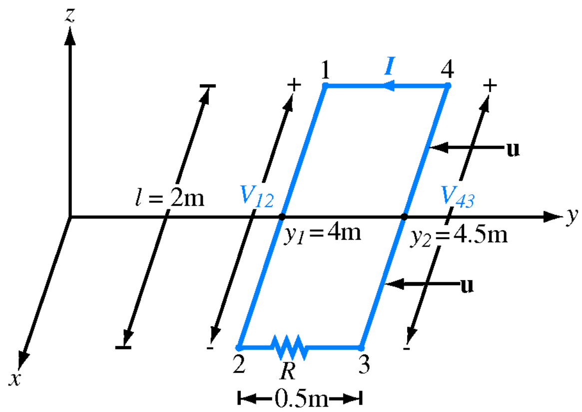
$$
\begin{aligned}
\Phi &= \int_S \mathbf{B} \cdot d\mathbf{s} \\
&= \int_{x=-0.1}^{0.1} \int_{y=-0.1}^{0.1} (\hat{\mathbf{z}}\, 100 x^2 \cos 10^3 t) \cdot \hat{\mathbf{z}}\, dx\, dy \\
&= (100 \cos 10^3 t) \times 0.2 \int_{-0.1}^{0.1} x^2\, dx \\
&= 20 \cos 10^3 t \left. \frac{x^3}{3} \right|_{-0.1}^{0.1} \\
&= \frac{20}{3} \cos 10^3 t ((0.1)^3 + (0.1)^3) = 13.3 \times 10^{-3} \cos 10^3 t.
\end{aligned}
$$

$$
\begin{aligned}
I &= \frac{V_{\text{emf}}}{R} \\
&= -\frac{N}{R} \frac{d\Phi}{dt} \\
&= -\frac{10}{1000} \frac{d}{dt} (13.3 \times 10^{-3} \cos 10^3 t) = 133 \sin 10^3 t \quad \text{(mA)}.
\end{aligned}
$$

At $t = 0$, $d\Phi/dt < 0$ and $V_{\text{emf}} > 0$. Since the flux is decreasing, Lenz's law requires I to be in the direction opposite that shown in the figure so that the flux inducted by I is in opposition to the trend of $d\Phi/dt$. Hence, in terms of the indicated direction of I,

$$
I = -133 \sin 10^3 t \quad \text{(mA)}.
$$

Exercise 6.3 For the moving loop of Fig. 6-9, find I when the loop sides are at $y_1 = 4$ m and $y_2 = 4.5$ m. Also, reverse the direction of motion such that $\mathbf{u} = -\hat{\mathbf{y}}5$ (m/s).



Solution: At $y_1 = 4$ m and $y_2 = 4.5$ m,

$$\mathbf{B}(y_1) = \hat{\mathbf{z}}0.2e^{-0.1\times 4} = \hat{\mathbf{z}}0.1340 \quad \text{(T)}.$$
$$\mathbf{B}(y_2) = \hat{\mathbf{z}}0.2e^{-0.1\times 4.5} = \hat{\mathbf{z}}0.1275 \quad \text{(T)}.$$

$$\begin{aligned} V_{12} &= \int_2^1 [\mathbf{u} \times \mathbf{B}(y_1)] \cdot dl \\ &= \int_{l/2}^{-l/2} (-\hat{\mathbf{y}}5 \times \hat{\mathbf{z}}0.134) \cdot \hat{\mathbf{x}}\, dx \\ &= 0.67l = 0.67 \times 2 = 1.340 \quad \text{(V)}. \end{aligned}$$

$$V_{43} = u\,B(y_2)l = 5 \times 0.1275 \times 2 = 1.275 \quad \text{(V)}$$

$$I = \frac{V_{43} - V_{12}}{R} = \frac{1.275 - 1.340}{5} = -13 \quad \text{(mA)}.$$

Exercise 6.4 Suppose that we turn the loop of Fig. 6-9 so that its surface is parallel to the x–z plane. What would I be in that case?

Solution: $I = 0$, because in that case V_{12} and V_{43} would always be equal because both are always at the same value of y, and hence the **B** field is the same for both of them.

Exercise 6.5 A poor conductor is characterized by a conductivity $\sigma = 100$ (S/m) and permittivity $\varepsilon = 4\varepsilon_0$. At what angular frequency ω is the amplitude of the conduction current density $\mathbf{J}$ equal to the amplitude of the displacement current density $\mathbf{J}_\text{d}$?

Solution:

$$|\mathbf{J}| = \sigma|\mathbf{E}|$$

$$|\mathbf{J}_\text{d}| = \left|\frac{\partial \mathbf{D}}{\partial t}\right| = |j\omega\varepsilon\mathbf{E}| = \omega\varepsilon|\mathbf{E}|$$

If $|\mathbf{J}| = |\mathbf{J}_\text{d}|$, then $\sigma = \omega\varepsilon$, or

$$\omega = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\varepsilon_\text{r}\varepsilon_0} = \frac{100}{4 \times 8.85 \times 10^{-12}} = 2.82 \times 10^{12} \quad \text{(rad/s)}.$$

Exercise 6.6 Determine (a) the relaxation time constant and (b) the time it takes for a charge density to decay to 1% of its initial value in quartz, given that $\varepsilon_r = 5$ and $\sigma = 10^{-17}$ S/m.

Solution:

(a)

$$\tau_r = \frac{\varepsilon}{\sigma} = \frac{\varepsilon_r \varepsilon_0}{\sigma} = \frac{5 \times 8.85 \times 10^{-12}}{10^{-17}} = 4.425 \times 10^6 \text{ s} = 51.2 \text{ days.}$$

(b)

$$\begin{aligned}
\rho_v(t) &= \rho_{v0} e^{-t/\tau_r} \\
\frac{\rho_v}{\rho_{v0}} &= 0.01 = e^{-t/51.2} \\
\ln 0.01 &= -\frac{t}{51.2} \\
t &= -51.2 \ln 0.01 = 236 \text{ days.}
\end{aligned}$$

Exercise 6.7 The magnetic field intensity of an electromagnetic wave propagating in a lossless medium with $\varepsilon = 9\varepsilon_0$ and $\mu = \mu_0$ is given by

$$\mathbf{H}(z,t) = \hat{\mathbf{x}}\,0.3\cos(10^8 t - kz + \pi/4) \quad \text{(A/m)}.$$

Find $\mathbf{E}(z,t)$ and k.

Solution:

$$\begin{aligned} k = \omega\sqrt{\mu\varepsilon} &= \omega\sqrt{\varepsilon_\mathrm{r}}\sqrt{\varepsilon_0\mu_0} \\ &= \frac{\omega\sqrt{\varepsilon_\mathrm{r}}}{c} = \frac{10^8 \times 3}{3 \times 10^8} = 1 \quad \text{(rad/m)}. \end{aligned}$$

$$\begin{aligned} \mathbf{H}(z,t) &= \hat{\mathbf{x}}\,0.3\cos(10^8 t - kz + \pi/4) \quad \text{(A/m)} \\ \widetilde{\mathbf{H}}(z) &= \hat{\mathbf{x}}\,0.3e^{-jkz}e^{j\pi/4} \end{aligned}$$

$$\begin{aligned} \nabla \times \widetilde{\mathbf{H}} &= j\omega\varepsilon\widetilde{\mathbf{E}} \\ \widetilde{\mathbf{E}} &= \frac{1}{j\omega\varepsilon}\,\nabla \times \widetilde{\mathbf{H}} \\ &= \frac{1}{j\omega\varepsilon}\,\hat{\mathbf{y}}\,\frac{\partial}{\partial z}(\widetilde{H}_x) \\ &= \frac{1}{j\omega\varepsilon}\,\hat{\mathbf{y}}\,\frac{\partial}{\partial z}(0.3e^{-jkz}e^{j\pi/4}) \\ &= -\hat{\mathbf{y}}\,\frac{0.3k}{\omega\varepsilon}\,e^{-jkz}e^{j\pi/4} \\ &= -\hat{\mathbf{y}}\,\frac{0.3 \times 1}{10^8 \times 8.85 \times 10^{-12} \times 9}\,e^{-jkz}e^{j\pi/4} \\ &= -\hat{\mathbf{y}}\,37.7e^{-jkz}e^{j\pi/4} \end{aligned}$$

$$\begin{aligned} \mathbf{E}(z,t) &= \mathfrak{Re}[\widetilde{\mathbf{E}}(z)\,e^{j\omega t}] \\ &= -\hat{\mathbf{y}}\,37.7\cos(10^8 t - z + \pi/4) \quad \text{(V/m)}. \end{aligned}$$

Chapter 7 Exercise Solutions

Exercise 7.1 A 10-MHz uniform plane wave is traveling in a nonmagnetic medium with $\mu = \mu_0$ and $\varepsilon_r = 9$. Find (a) the phase velocity, (b) the wavenumber, (c) the wavelength in the medium, and (d) the intrinsic impedance of the medium.

Solution:

(a)

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times 10^8}{\sqrt{9}} = 10^8 \quad \text{(m/s)}.$$

(b)

$$k = \frac{\omega}{u_p} = \frac{2\pi\times 10^7}{10^8} = 0.2\pi \quad \text{(rad/m)}.$$

(c)

$$k = \frac{2\pi}{\lambda}, \qquad \lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2\pi} = 10 \quad \text{(m)}.$$

(d)

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot\frac{1}{\sqrt{\varepsilon_r}} = \frac{377}{3} = 125.67\ \Omega.$$

Exercise 7.2 The electric field phasor of a uniform plane wave traveling in a lossless medium with an intrinsic impedance of 188.5 Ω is given by $\widetilde{\mathbf{E}} = \hat{\mathbf{z}}\,10e^{-j4\pi y}$ (mV/m). Determine (a) the associated magnetic field phasor and (b) the instantaneous expression for $\mathbf{E}(y,t)$ if the medium is nonmagnetic ($\mu = \mu_0$).

Solution:

(a) $\mathbf{E}$ is along $+\hat{\mathbf{z}}$ and the wave direction $\hat{\mathbf{k}}$ is along $+\hat{\mathbf{y}}$. It follows that $\widetilde{\mathbf{H}}$ is along $\hat{\mathbf{k}} \times \mathbf{E} = \hat{\mathbf{y}} \times \hat{\mathbf{z}}E = \hat{\mathbf{x}}E$. Hence,

$$\widetilde{\mathbf{H}} = \hat{\mathbf{x}}\,\frac{10 \times 10^{-3}}{188.5}\,e^{-j4\pi y} = \hat{\mathbf{x}}\,53e^{-j4\pi y} \quad (\mu\text{A/m}).$$

(b) From the given coefficient of y,

$$k = 4\pi \quad \text{(rad/m)}.$$

From $\eta = \eta_0/\sqrt{\varepsilon_r}$,

$$\varepsilon_r = \left(\frac{\eta_0}{\eta}\right)^2 = \left(\frac{377}{188.5}\right)^2 = 4.$$

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{2} = 1.5 \times 10^8 \quad \text{(m/s)}.$$

$$\omega = ku_p = 4\pi \times 1.5 \times 10^8 = 6\pi \times 10^8 \quad \text{(rad/s)}.$$

$$\begin{aligned}\mathbf{E}(z,t) &= \mathfrak{Re}[\widetilde{\mathbf{E}}(z)\,e^{j\omega t}] \\ &= \hat{\mathbf{z}}\,10\cos(\omega t - ky) \\ &= \hat{\mathbf{z}}\,10\cos(6\pi \times 10^8 t - 4\pi y) \quad \text{(mV/m)}.\end{aligned}$$

Exercise 7.3 If the magnetic field phasor of a plane wave traveling in a medium with intrinsic impedance $\eta = 100\ \Omega$ is given by $\widetilde{\mathbf{H}} = (\hat{\mathbf{y}}\,10 + \hat{\mathbf{z}}\,20)e^{-j4x}$ (mA/m), find the associated electric field phasor.

Solution:

$$\widetilde{\mathbf{H}} = (\hat{\mathbf{y}}\,10 + \hat{\mathbf{z}}\,20)e^{-j4x} \quad \text{(mA/m)}$$

The phase factor of $\widetilde{\mathbf{H}}$ denotes that $\hat{\mathbf{k}} = \hat{\mathbf{x}}$.

$$\begin{aligned}
\widetilde{\mathbf{E}} &= -\eta\hat{\mathbf{k}} \times \widetilde{\mathbf{H}} \\
&= -100[\hat{\mathbf{x}} \times (\hat{\mathbf{y}}\,10 + \hat{\mathbf{z}}\,20)]e^{-j4x} \times 10^{-3} \\
&= (-\hat{\mathbf{z}} + \hat{\mathbf{y}}\,2)e^{-j4x} \quad \text{(V/m)}.
\end{aligned}$$

Exercise 7.4 Repeat Exercise 7.3 for a magnetic field given by

$$\widetilde{\mathbf{H}} = \hat{\mathbf{y}}(10e^{-j3x} - 20e^{j3x}) \quad \text{(mA/m)}.$$

Solution:

$$\widetilde{\mathbf{H}} = \hat{\mathbf{y}}\,(10e^{-j3x} - 20e^{j3x}) \quad \text{(mA/m)}$$

This magnetic field is composed of two components, one with amplitude of 10 (mA/m) belonging to a wave traveling along $+\hat{\mathbf{x}}$ and another with amplitude of 20 (mA/m) belonging to a separate wave traveling in the opposite direction ($-\hat{\mathbf{x}}$). Hence, we need to treat these two components separately:

$$\widetilde{\mathbf{H}} = \widetilde{\mathbf{H}}_1 + \widetilde{\mathbf{H}}_2$$

with

$$\begin{aligned} \widetilde{\mathbf{H}}_1 &= \hat{\mathbf{y}}\,10e^{-j3x} \quad \text{(mA/m)}, \\ \widetilde{\mathbf{H}}_2 &= -\hat{\mathbf{y}}\,20e^{-j3x} \quad \text{(mA/m)}. \end{aligned}$$

For the first wave:

$$\begin{aligned} \widetilde{\mathbf{E}}_1 &= -\eta\hat{\mathbf{k}} \times \widetilde{\mathbf{H}}_1 \\ &= -100(\hat{\mathbf{x}} \times \hat{\mathbf{y}}\,10e^{-j3x}) = -\hat{\mathbf{z}}\,e^{-j3x} \quad \text{(V/m)}. \end{aligned}$$

For the second wave:

$$\begin{aligned} \widetilde{\mathbf{E}}_2 &= -100[-\hat{\mathbf{x}} \times (-\hat{\mathbf{y}}\,20e^{j3x})] \\ &= -\hat{\mathbf{z}}\,2e^{j3x} \quad \text{(V/m)}. \end{aligned}$$

$$\begin{aligned} \widetilde{\mathbf{E}} &= \widetilde{\mathbf{E}}_1 + \widetilde{\mathbf{E}}_2 \\ &= -\hat{\mathbf{z}}\,(e^{-j3x} + 2e^{j3x}) \quad \text{(V/m)}. \end{aligned}$$

Exercise 7.5 The electric field of a plane wave is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 3\cos(\omega t - kz) + \hat{\mathbf{y}}\, 4\cos(\omega t - kz) \quad \text{(V/m)}.$$

Determine (a) the polarization state, (b) the modulus of **E**, and (c) the inclination angle.

Solution:

(a) Since the x- and y-components are exactly in phase with each other, the polarization state has to be linear. This can also be ascertained by verifying that $\chi = 0$. From the given expression for $\mathbf{E}(z,t)$, $\delta_x = \delta_y = 0$. Hence, $\delta = \delta_y - \delta_x = 0$.

$$\begin{aligned} \psi_0 &= \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\frac{4}{3} = 53.1^\circ. \\ \sin 2\chi &= (\sin 2\psi_0)\sin\delta = 0 \\ \chi &= 0. \end{aligned}$$

(b) $|\mathbf{E}| = [E_x^2 + E_y^2]^{1/2} = 5\cos(\omega t - kz)$ (V/m).

(c) From part (a), $\psi_0 = 53.1^\circ$.

Exercise 7.6 If the electric field phasor of a TEM wave is given by $\widetilde{\mathbf{E}} = (\hat{\mathbf{y}} - \hat{\mathbf{z}}j)e^{-jkx}$, determine the polarization state.

Solution:

$$-j = e^{-j\pi} \cdot e^{j\pi/2} = e^{-j\pi/2}$$

Hence,

$$\begin{aligned}\widetilde{\mathbf{E}} &= (\hat{\mathbf{y}} - \hat{\mathbf{z}}\,j)e^{-jkx} \\ &= \hat{\mathbf{y}} + \hat{\mathbf{z}}\,e^{-jkx}e^{-j\pi/2}\end{aligned}$$

It follows that

$$\delta_y = 0, \qquad \delta_z = -\frac{\pi}{2}\,.$$

$$\delta = \delta_z - \delta_y = -\frac{\pi}{2}\,.$$

Since $a_y = a_z = 1$ and $\delta = -\pi/2$, the wave is RHC.

Exercise 7.7 The constitutive parameters of copper are $\mu = \mu_0 = 4\pi \times 10^{-7}$ (H/m), $\varepsilon = \varepsilon_0 \simeq (1/36\pi) \times 10^{-9}$ (F/m), and $\sigma = 5.8 \times 10^7$ (S/m). Assuming that these parameters are frequency independent, over what frequency range of the electromagnetic spectrum [see Fig. 1-16] is copper a good conductor?

Solution: Good conductor implies that

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} > 100$$

or

$$\omega = 2\pi f < \frac{\sigma}{100\varepsilon}$$
$$f < \frac{\sigma}{200\pi\varepsilon} = \frac{5.8 \times 10^7}{200\pi \times (1/36\pi) \times 10^{-9}} = 1.04 \times 10^{16} \text{ Hz.}$$

Exercise 7.8 Over what frequency range may dry soil, with $\varepsilon_r = 3$, $\mu_r = 1$, and $\sigma = 10^{-4}$ (S/m), be regarded as a low-loss dielectric medium?

Solution: Low loss dielectric implies that

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} < 0.01$$

or

$$\omega = 2\pi f > \frac{\sigma}{0.01\varepsilon}$$
$$f > \frac{100\sigma}{2\pi\varepsilon_r\varepsilon_0} = \frac{100 \times 10^{-4}}{2\pi \times 3 \times (1/36\pi) \times 10^{-9}} = 60 \text{ MHz.}$$

Exercise 7.9 For a wave traveling in a medium with a skin depth δ_s, what is the amplitude of $\mathbf{E}$ at a distance of $3\delta_s$ compared with its initial value?

Solution: For a wave traveling in the z-direction:

$$|E(z)| = |E_0|e^{-\alpha z} = |E_0|e^{-z/\delta_s}$$

where we used the fact that $\delta_s = 1/\alpha$. At $z = 3\delta_s$,

$$\frac{|E(z = 3\delta_s)|}{|E_0|} = e^{-3} = 0.05 = 5\%.$$

Exercise 7.10 Convert the following values of the power ratio G to decibels: (a) 2.3, (b) 4×10^3, (c) 3×10^{-2}.

Solution:

(a) $10 \log 2.3 = 3.6$ dB.

(b) $10 \log 4 \times 10^3 = 10 \log 4 + 10 \log 10^3 = 6 + 30 = 36$ dB.

(c) $10 \log 3 \times 10^{-2} = 10 \log 3 + 10 \log 10^{-2} = 4.8 - 20 = -15.2$ dB.

Exercise 7.11 Find the voltage ratio g in natural units corresponding to the following decibel values of the power ratio G: (a) 23 dB, (b) -14 dB, (c) -3.6 dB.

Solution:

(a)

$$\begin{aligned}
10\log G &= 23 \text{ dB} \\
\log G &= \frac{23}{10} = 2.3 \\
G &= 10^{2.3} = 199.53 \\
g &= \sqrt{G} = \sqrt{199.53} = 14.13.
\end{aligned}$$

(b)

$$\begin{aligned}
10\log G &= -14 \text{ dB} \\
\log G &= -1.4 \\
G &= 10^{-1.4} = 0.04 \\
g &= \sqrt{G} = 0.2.
\end{aligned}$$

(c)

$$\begin{aligned}
10\log G &= -3.6 \text{ dB} \\
\log G &= -0.36 \\
G &= 10^{-0.36} = 0.436 \\
g &= \sqrt{G} = 0.66.
\end{aligned}$$

Chapter 8 Exercise Solutions

Exercise 8.1 To eliminate reflections of normally incident plane waves, a dielectric slab of thickness d and relative permittivity ε_{r_2} is to be inserted between two semi-infinite media with relative permittivities $\varepsilon_{r_1} = 1$ and $\varepsilon_{r_3} = 16$. Use the quarter-wave transformer technique to select d and ε_{r_2}. Assume $f = 3$ GHz.

Solution: The quarter-wave transformer technique requires that the in-between layer be $\lambda/4$ thick (as well as $\lambda/4$ plus multiples of $\lambda/2$) and that its characteristic impedance be related to the impedances of the two media as follows:

$$\eta^2 = \sqrt{\eta_1 \eta_2}\,.$$

Thus,

$$\frac{\mu_0}{\varepsilon_r \varepsilon_0} = \sqrt{\frac{\mu_0}{\varepsilon_{r1}\varepsilon_0}\frac{\mu_0}{\varepsilon_{r2}\varepsilon_0}}$$

or

$$\varepsilon_r = \sqrt{\varepsilon_{r1}\varepsilon_{r2}} = \sqrt{1 \times 16} = 4.$$

At 3 GHz,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{c}{f\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{3 \times 10^9 \sqrt{4}} = 5 \text{ cm}.$$

Hence,

$$d = \frac{\lambda}{4} + \frac{n\lambda}{4} = (1.25 + 2.5n) \quad \text{(cm)}.$$

Exercise 8.2 Express the normal-incidence reflection coefficient at the boundary between two nonmagnetic, conducting media in terms of their complex permittivities.

Solution: From (8.24a):

$$\Gamma = \frac{\eta_{c2} - \eta_{c1}}{\eta_{c2} + \eta_{c1}}$$
$$= \frac{\sqrt{\dfrac{\mu_0}{\varepsilon_{c2}}} - \sqrt{\dfrac{\mu_0}{\varepsilon_{c1}}}}{\sqrt{\dfrac{\mu_0}{\varepsilon_{c2}}} + \sqrt{\dfrac{\mu_0}{\varepsilon_{c1}}}} = \frac{\sqrt{\varepsilon_{c1}} - \sqrt{\varepsilon_{c2}}}{\sqrt{\varepsilon_{c1}} + \sqrt{\varepsilon_{c2}}}\,.$$

Exercise 8.3 Obtain expressions for the average power densities in media 1 and 2 for the fields described by Eqs. (8.22a) through (8.23b), assuming medium 1 is slightly lossy with η_{c_1} approximately real.

Solution: With $\gamma_1 = \alpha_1 + j\beta_1$,

$$\begin{aligned}
\widetilde{\mathbf{E}}_1(z) &= \hat{\mathbf{x}} E_0^{\mathrm{i}} (e^{-(\alpha_1+j\beta_1)z} + \Gamma e^{(\alpha_1+j\beta_1)z}) \\
\widetilde{\mathbf{H}}_1^*(z) &= \hat{\mathbf{y}} \frac{(E_0^{\mathrm{i}})^*}{\eta_{c1}^*} (e^{-(\alpha_1-j\beta_1)z} - \Gamma^* e^{(\alpha_1+j\beta_1)z}). \\
\mathbf{S}_{\mathrm{av}_1} &= \frac{1}{2} \mathfrak{Re} \left\{ \widetilde{\mathbf{E}}_1 \times \widetilde{\mathbf{H}}_1^* \right\} \\
&= \hat{\mathbf{z}} \frac{1}{2} \mathfrak{Re} \left\{ \frac{|E_0^{\mathrm{i}}|^2}{\eta_{c1}^*} [e^{-2\alpha_1 z} - |\Gamma|^2 e^{2\alpha_1 z} - \Gamma^* e^{-j2\beta_1 z} + \Gamma e^{j2\beta_1 z}] \right\}
\end{aligned}$$

For η_{c1} approximately real, $\eta_{c1}^* \approx \eta_c$,

$$\mathbf{S}_{\mathrm{av}_1} = \hat{\mathbf{z}} \frac{|E_0^{\mathrm{i}}|^2}{2\eta_{c1}} \left\{ (e^{-2\alpha_1 z} - |\Gamma|^2 e^{2\alpha_1 z}) + \mathfrak{Re}[-\Gamma^* e^{-j2\beta_1 z} + \Gamma e^{j2\beta_1 z}] \right\}$$

With $\Gamma = |\Gamma| e^{j\theta_{\mathrm{r}}}$ and $\psi = 2\beta_1 z + \theta_{\mathrm{r}}$, let us consider the last two terms:

$$\begin{aligned}
\mathfrak{Re}[-\Gamma^* e^{-j2\beta_1 z} + \Gamma e^{j2\beta_1 z}] &= \mathfrak{Re}[-|\Gamma| e^{-j\psi} + |\Gamma| e^{j\psi}] \\
&= \mathfrak{Re}[-|\Gamma| \cos\psi + j|\Gamma| \sin\psi + |\Gamma| \cos\psi + j|\Gamma| \sin\psi] \\
&= \mathfrak{Re}[j2|\Gamma| \sin\psi] \\
&= 0.
\end{aligned}$$

Hence,

$$\mathbf{S}_{\mathrm{av}_1} = \hat{\mathbf{z}} \frac{|E_0^{\mathrm{i}}|^2}{2\eta_{c1}} (e^{-2\alpha_1 z} - |\Gamma|^2 e^{2\alpha_1 z}).$$

For the wave transmitted into medium 2,

$$\begin{aligned}
\mathbf{S}_{\mathrm{av}_2} &= \frac{1}{2} \mathfrak{Re}[\widetilde{\mathbf{E}}_1 \times \widetilde{\mathbf{H}}_2^*] \\
&= \frac{1}{2} \mathfrak{Re} \left[\hat{\mathbf{x}}\, \tau E_0^{\mathrm{i}} e^{-(\alpha_2+j\beta_2)z} \times \hat{\mathbf{y}}\, \tau^* \frac{(E_0^{\mathrm{i}})^*}{\eta_{c2}^*}\, e^{-(\alpha_2-j\beta_2)z} \right] \\
&= \hat{\mathbf{z}} |\tau|^2 \frac{|E_0^{\mathrm{i}}|^2}{2}\, e^{-2\alpha_2 z}\, \mathfrak{Re} \left(\frac{1}{\eta_{c2}^*} \right).
\end{aligned}$$

Exercise 8.4 In the visible part of the electromagnetic spectrum, the index of refraction of water is 1.33. What is the critical angle for light waves generated by an upward-looking underwater light source?

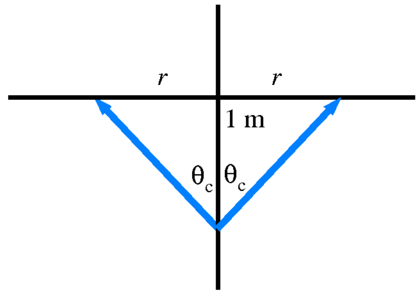
Solution:

$$\sin\theta_c = \frac{n_2}{n_1} = \frac{1}{1.33}\,.$$

$$\theta_c = \sin^{-1}\left(\frac{1}{1.33}\right) = 48.8^\circ.$$

Exercise 8.5 If the light source of Exercise 8.4 is situated at a depth of 1 m below the water surface and if its beam is isotropic (radiates in all directions), how large a circle would it illuminate when observed from above?

Solution:



$$\begin{aligned} r &= 1 \text{ m} \times \tan\theta_c \\ &= \tan 48.8^\circ = 1.14 \text{ m} \end{aligned}$$

$$d = 2r = 2.28 \text{ m}.$$

Exercise 8.6 If the index of refraction of the cladding material in Example 8-5 is increased to 1.50, what would be the new maximum usable data rate?

Solution:

$$\begin{aligned} f_{\mathrm{p}} &= \frac{cn_{\mathrm{c}}}{2ln_{\mathrm{f}}(n_{\mathrm{f}}-n_{\mathrm{c}})} \\ &= \frac{3\times10^{8}\times1.50}{2\times10^{3}\times1.52(1.52-1.50)} = 7.4 \quad \text{(Mb/s)}. \end{aligned}$$

Exercise 8.7 A wave in air is incident upon a soil surface at $\theta_i = 50°$. If soil has $\varepsilon_r = 4$ and $\mu_r = 1$, determine $\Gamma_\perp$, $\tau_\perp$, $\Gamma_\parallel$, and $\tau_\parallel$.

Solution: Using (8.60),

$$\begin{aligned}
\Gamma_\perp &= \frac{\cos\theta_i - \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}} \\
&= \frac{\cos 50° - \sqrt{4 - \sin^2 50°}}{\cos 50° + \sqrt{4 - \sin^2 50°}} = -0.48. \\
\tau_\perp &= 1 + \Gamma_\perp = 1 - 0.48 = 0.52.
\end{aligned}$$

Using (8.68),

$$\begin{aligned}
\Gamma_\parallel &= \frac{-(\varepsilon_2/\varepsilon_1)\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}}{(\varepsilon_2/\varepsilon_1)\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}} \\
&= \frac{-4\cos 50° + \sqrt{4 - \sin^2 50°}}{4\cos 50° + \sqrt{4 - \sin^2 50°}} = -0.16.
\end{aligned}$$

$$\begin{aligned}
\sin\theta_t &= \frac{n_1}{n_2}\sin\theta_i = \frac{1}{\sqrt{4}}\sin 50° = 0.383 \\
\theta_t &= 22.5°. \\
\tau_\parallel &= (1 + \Gamma_\parallel)\frac{\cos\theta_i}{\cos\theta_t} = (1 - 0.16)\frac{\cos 50°}{\cos 22.5°} = 0.58.
\end{aligned}$$

Exercise 8.8 Determine the Brewster angle for the boundary of Exercise 8.7.

Solution:

$$\theta_{\mathrm{B}} = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \tan^{-1}\sqrt{4} = 63.4^\circ.$$

Exercise 8.9 Show that the incident, reflected, and transmitted electric and magnetic fields given by Eqs. (8.65a) through (8.65f) all have the same exponential phase function along the x-direction.

Solution: With the help of Snell's laws of reflection and refraction (Eqs. (8.55) and (8.56)), the x-components of the exponential phase functions are:

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{i}}_{\|} &\sim e^{-jk_1 x \sin\theta_{\mathrm{i}}} \\
\widetilde{\mathbf{H}}^{\mathrm{i}}_{\|} &\sim e^{-jk_1 x \sin\theta_{\mathrm{i}}} \\
\widetilde{\mathbf{E}}^{\mathrm{r}}_{\|} &\sim e^{-jk_1 x \sin\theta_{\mathrm{r}}} = e^{-jk_1 x \sin\theta_{\mathrm{i}}}, \qquad (\theta_{\mathrm{i}} = \theta_{\mathrm{r}}) \\
\widetilde{\mathbf{H}}^{\mathrm{r}}_{\|} &\sim e^{-jk_1 x \sin\theta_{\mathrm{r}}} = e^{-jk_1 x \sin\theta_{\mathrm{i}}}, \qquad (\theta_{\mathrm{i}} = \theta_{\mathrm{r}}) \\
\widetilde{\mathbf{E}}^{\mathrm{t}}_{\|} &\sim e^{-jk_2 x \sin\theta_{\mathrm{t}}} = e^{-jk_1 x \sin\theta_{\mathrm{i}}}, \qquad k_2 \sin\theta_{\mathrm{t}} = k_1 \sin\theta_{\mathrm{i}} \\
\widetilde{\mathbf{H}}^{\mathrm{t}}_{\|} &\sim e^{-jk_2 x \sin\theta_{\mathrm{t}}} = e^{-jk_1 x \sin\theta_{\mathrm{i}}}, \qquad k_2 \sin\theta_{\mathrm{t}} = k_1 \sin\theta_{\mathrm{i}}.
\end{aligned}$$

Exercise 8.10 For a square waveguide with $a = b$, what is the value of the ratio $\widetilde{E}_x/\widetilde{E}_y$ for the TM_{11} mode?

Solution: According to the expressions for $\widetilde{E}_x$ and $\widetilde{E}_y$ given by Eqs. (8.104a and b),

$$\frac{\widetilde{E}_x}{\widetilde{E}_y} = \frac{\tan(\pi y/a)}{\tan(\pi x/a)} \qquad \text{for TM}_{11}.$$

Exercise 8.11 What is the cutoff frequency for the dominant TM mode in a waveguide filled with a material with $\varepsilon_r = 4$? The waveguide dimensions are $a = 2b = 5$ cm.

Solution: For TM_{11}, Eq. (8.106) gives

$$f_{11} = \frac{3\times10^8}{2\sqrt{4}}\left[\left(\frac{1}{5\times10^{-2}}\right)^2 + \left(\frac{1}{2.5\times10^{-2}}\right)^2\right]^{1/2}$$
$$= 3.35\times10^9 \text{ Hz} = 3.35 \text{ GHz}.$$

Exercise 8.12 What is the magnitude of the phase velocity of a TE or TM mode at $f = f_{mn}$?

Solution: According to Eq. (8.108), at $f = f_{mn}$, $u_\text{p} = \infty$.

Exercise 8.13 What do the wave impedances for TE and TM look like as f approaches f_{mn}?

Solution: According to Eqs. (8.109) and (8.111), at $f = f_{mn}$, Z_{TE} becomes infinite and Z_{TM} becomes zero, resembling an open circuit and short circuit, respectively.

Exercise 8.14 What are the values for (a) u_p, (b) u_g, and (c) the zigzag angle θ' at $f = 2f_{10}$ for a TE$_{10}$ mode in a hollow waveguide?

Solution: For a hollow waveguide,

$$u_\mathrm{p} = \frac{c}{\sqrt{1-(f_{10}/2f_{10})^2}} = \frac{c}{\sqrt{1-1/4}} = 1.15c.$$

$$u_\mathrm{g} = c\sqrt{1-1/4} = 0.87c.$$

$$\theta' = \tan^{-1}\left(\frac{1}{\sqrt{(2f_{10}/f_0)^2-1}}\right) = 30^\circ.$$

Chapter 9 Exercise Solutions

Exercise 9.1 A 1-m-long dipole is excited by a 5-MHz current with an amplitude of 5 A. At a distance of 2 km, what is the power density radiated by the antenna along its broadside direction?

Solution: At 5 MHz,

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{5\times 10^6} = 60 \text{ m}.$$

From (9.14),

$$\begin{aligned} S_0 &= \frac{15\pi I_0^2}{R^2}\left(\frac{l}{\lambda}\right)^2 \\ &= \frac{15\pi 5^2}{(2\times 10^3)^2}\times\left(\frac{1}{60}\right)^2 = 8.2\times 10^{-8} \text{ W/m}^2. \end{aligned}$$

Exercise 9.2 An antenna has a conical radiation pattern with a normalized radiation intensity $F(\theta) = 1$ for θ between 0° and 45° and zero intensity for θ between 45° and 180°. The pattern is independent of the azimuth angle ϕ. Find (a) the pattern solid angle and (b) the directivity.

Solution:

(a) From (9.21),

$$\begin{aligned}
\Omega_\text{p} &= \iint_{4\pi} F(\theta,\phi)\, d\Omega \\
&= \int_{\theta=0}^{45^\circ} \int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi \\
&= -2\pi \cos\theta \big|_0^{45^\circ} = -2\pi(\cos 45^\circ - \cos 0^\circ) = 1.84 \text{ sr}.
\end{aligned}$$

(b)

$$D = \frac{4\pi}{\Omega_\text{p}} = \frac{4\pi}{1.84} = 6.83.$$

Exercise 9.3 The maximum power density radiated by a short dipole at a distance of 1 km is 60 (nW/m^2). If $I_0 = 10$ A, find the radiation resistance.

Solution: From (9.14),

$$S_0 = \frac{15\pi I_0^2}{R^2}\left(\frac{l}{\lambda}\right)^2$$

and from (9.35),

$$R_{\text{rad}} = 80\pi^2\left(\frac{l}{\lambda}\right)^2.$$

Hence,

$$\begin{aligned} R_{\text{rad}} &= 80\pi^2\left(\frac{S_0R^2}{15\pi I_0^2}\right) \\ &= \frac{80\pi}{15}\frac{S_0R^2}{I_0^2} \\ &= \frac{80\pi}{15}\times\frac{60\times10^{-9}\times(10^3)^2}{10^2} = 10^{-2}\ \Omega = 10\ \text{m}\Omega. \end{aligned}$$

Exercise 9.4 For the half-wave dipole antenna, evaluate $F(\theta)$ versus θ in order to determine the half-power beamwidth in the elevation plane (the plane containing the dipole axis).

Solution: From (9.44),

$$F(\theta) = \left[\frac{\cos[(\pi/2)\cos\theta]}{\sin\theta}\right]^2.$$

To find θ at which $F(\theta) = 0.5$,

$$\frac{\cos[(\pi/2)\cos\theta]}{\sin\theta} = (0.5)^{1/2} = 0.707.$$

Because this equation has no straightforward way for solving for θ, an alternative approach is to calculate $F(\theta)$ versus θ numerically and to note the angle at which $F(\theta) = 0.5$. Such an approach leads to $\theta = 39°$. The beamwidth is twice this value,

$$\beta = 78°.$$

Exercise 9.5 If the maximum power density radiated by a half-wave dipole is 50 (μW/m^2) at a range of 1 km, what is the current amplitude I_0?

Solution: From

$$S_{\max} = \frac{15 I_0^2}{\pi R^2} ,$$

$$I_0 = \sqrt{\frac{\pi R^2}{15} \, S_{\max}} = \left[\frac{\pi \times (10^3)^2 \times 50 \times 10^{-6}}{15} \right]^{1/2} = 3.24 \text{ A.}$$

Exercise 9.6 The effective area of an antenna is 9 m^2. What is its directivity in decibels at 3 GHz?

Solution: At 3 GHz,

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^9} = 0.1 \text{ m}$$

$$D = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi \times 9}{(0.1)^2} = 11310 = 40.53 \text{ dB}.$$

Exercise 9.7 At 100 MHz, the pattern solid angle of an antenna is 1.3 sr. Find (a) the antenna directivity D and (b) its effective area A_e.

Solution: At 100 MHz,

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^8} = 3 \text{ m}$$

(a)

$$D = \frac{4\pi}{\Omega_p} = \frac{4\pi}{1.3} = 9.67.$$

(b)

$$A_e = \frac{\lambda^2 D}{4\pi} = \frac{3^2 \times 9.67}{4\pi} = 6.92 \text{ m}^2.$$

Exercise 9.8 If the operating frequency of the communication system described in Example 9-4 is doubled to 12 GHz, what would then be the minimum required diameter of a home receiving TV antenna?

Solution: P_n remains the same, but with all other parameters remaining the same, P_rec will increase by a factor of 4 because P_rec is proportional to $1/\lambda^2$ and $\lambda = c/f$. This means that we can maintain P_rec the same by reducing A_r by a factor of 4, or equivalently by reducing d_r by a factor of 2 down to $2.55/2 = 1.27$ m.

Exercise 9.9 A 3-GHz microwave link consists of two identical antennas each with a gain of 30 dB. Determine the received power, given that the transmitter output power is 1 kW and the two antennas are 10 km apart.

Solution:

$$P_{\text{rec}} = P_{\text{t}} G_{\text{t}} G_{\text{r}} \left(\frac{\lambda}{4\pi R}\right)^2$$

$$P_{\text{t}} = 10^3 \text{ W}, \qquad G_{\text{t}} = G_{\text{r}} = 10^3 \text{ (30 dB)},$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^9} = 0.1 \text{ m}, \qquad R = 10^4 \text{ m}.$$

$$P_{\text{rec}} = 10^3 \times 10^6 \left(\frac{0.1}{4\pi \times 10^4}\right)^2 = 6.33 \times 10^{-4} \text{ W}.$$

Exercise 9.10 The effective area of a parabolic dish antenna is approximately equal to its physical aperture. If the directivity of a dish antenna is 30 dB at 10 GHz, what is its effective area? If the frequency is increased to 30 GHz, what will be its new directivity?

Solution: At 10 GHz,

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^9} = 0.03 \text{ m}.$$
$$A_e = \frac{\lambda^2 D}{4\pi} = \frac{0.03^2 \times 10^3}{4\pi} = 0.07 \text{ m}^2.$$

If f is increased to 30 GHz (by a factor of 3), λ becomes smaller by a factor of 3 and D larger by a factor of 9. Hence,

$$D = 9 \times 10^3 = 39.44 \text{ dB}.$$

Exercise 9.11 Verify that Eq. (9.86) is a solution of Eq. (9.85) by calculating $\text{sinc}^2 t$ for $t = 1.39$.

Solution: For $t = 1.39$ rad,

$$\text{sinc}^2 1.39 = \left(\frac{\sin 1.39}{1.39}\right)^2 = \left(\frac{0.98}{1.39}\right)^2 = 0.5.$$

Exercise 9.12 With its boresight direction along z, a square aperture was observed to have half-power beamwidths of 3° in both the x–z and y–z planes. Determine its directivity in decibels.

Solution: Using (9.96) and with

$$\beta = 3^\circ = \frac{3^\circ \times \pi}{180^\circ} = 0.0524 \text{ rad},$$

$$D = \frac{4\pi}{\beta_{xz}\beta_{yz}} = \frac{4\pi}{(0.0524)^2} = 4583.66 = 36.61 \text{ dB}.$$

Exercise 9.13 What condition must be satisfied in order to use scalar diffraction to compute the field radiated by an aperture antenna? Can we use it to compute the directional pattern of the eye's pupil ($d \cong 0.2$ cm) in the visible part of the spectrum ($\lambda = 0.35$ to $0.7\ \mu$m)? What would the beamwidth of the eye's directional pattern be at $\lambda = 0.5\ \mu$m?

Solution: Scalar diffraction is applicable when the aperture dimensions are much larger than λ, which certainly is the case for the eye's pupil ($d/\lambda = 0.2\text{ cm}/0.5\ \mu\text{m} = 4000$).

$$\begin{aligned}
\beta \approx \frac{\lambda}{d} = \left(\frac{1}{4000}\right) &= 2.5 \times 10^{-4} \text{ rad} \\
&= \frac{2.5 \times 10^{-4} \times 180}{\pi} \\
&= 0.0143^\circ \\
&= 0.86'
\end{aligned}$$

where $60' = 1^\circ$.

Exercise 9.14 Derive an expression for the array factor of a two-element array excited in phase with $a_0 = 1$ and $a_1 = 3$. The elements are positioned along the z-axis and are separated by $\lambda/2$.

Solution: Applying (9.110) with $a_0 = 1,\ a_1 = 3,\ \psi_0 = \psi_1 = 0$, and $d = \lambda/2$,

$$\begin{aligned}
F_{\text{a}}(\theta) &= \left|1 + 3e^{j(2\pi/\lambda)(\lambda/2)\cos\theta}\right|^2 \\
&= \left|1 + 3e^{j\pi\cos\theta}\right|^2 \\
&= (1 + 3e^{j\pi\cos\theta})(1 + 3e^{-j\pi\cos\theta}) \\
&= [1 + 9 + 3(e^{j\pi\cos\theta} + e^{-j\pi\cos\theta})] \\
&= [10 + 6\cos(\pi\cos\theta)].
\end{aligned}$$

Exercise 9.15 An equally spaced N-element array arranged along the z-axis is fed with equal amplitudes and phases; that is, $A_i = 1$ for $i = 0, 1, \ldots, N-1$. What is the magnitude of the array factor in the broadside direction?

Solution: Application of (9.107) with $A_i = 1$ for all i,

$$\begin{aligned} F_{\rm a}(\theta) &= \left| \sum_{i=0}^{N-1} e^{jid\cos\theta} \right|^2 \\ &= \left| 1 + e^{jkd\cos\theta} + e^{j2kd\cos\theta} + \cdots + e^{j(N-1)d\cos\theta} \right|^2 \end{aligned}$$

Along the broadside direction ($\theta = 90^\circ$),

$$\begin{aligned} F_{\rm a}(90^\circ) &= |1+1+1+\cdots+1|^2 \\ &= N^2. \end{aligned}$$

Chapter 10 Exercise Solutions

There are no exercises for Chapter 10.

